

MODULI SPACES AND FORMAL OPERADS

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ABSTRACT. Let $\overline{\mathcal{M}}_{g,l}$ be the moduli space of stable algebraic curves of genus g with l marked points. With the operations which relate the different moduli spaces identifying marked points, the family $(\overline{\mathcal{M}}_{g,l})_{g,l}$ is a modular operad of projective smooth Deligne-Mumford stacks, $\overline{\mathcal{M}}$. In this paper we prove that the modular operad of singular chains $C_*(\overline{\mathcal{M}}; \mathbb{Q})$ is formal; so it is weakly equivalent to the modular operad of its homology $H_*(\overline{\mathcal{M}}; \mathbb{Q})$. As a consequence, the “up to homotopy” algebras of these two operads are the same. To obtain this result we prove a formality theorem for operads analogous to Deligne-Griffiths-Morgan-Sullivan formality theorem, the existence of minimal models of modular operads, and a characterization of formality for operads which shows that formality is independent of the ground field.

1. INTRODUCTION

In recent years, moduli spaces of Riemann surfaces such as the moduli spaces of stable algebraic curves of genus g with l marked points, $\overline{\mathcal{M}}_{g,l}$, have played an important role in the mathematical formulation of certain theories inspired by physics, such as the complete cohomological field theories.

In these developments, the operations which relate the different moduli spaces $\overline{\mathcal{M}}_{g,l}$ identifying marked points, $\overline{\mathcal{M}}_{g,l} \times \overline{\mathcal{M}}_{h,m} \longrightarrow \overline{\mathcal{M}}_{g+h,l+m-2}$ and $\overline{\mathcal{M}}_{g,l} \longrightarrow \overline{\mathcal{M}}_{g+1,l-2}$, have been interpreted in terms of operads. With these operations the spaces $\overline{\mathcal{M}}_{0,l}$, $l \geq 3$, form a cyclic operad of projective smooth varieties, $\overline{\mathcal{M}}_0$ ([GeK94]), and the spaces $\overline{\mathcal{M}}_{g,l}$, $g, l \geq 0$, $2g - 2 + l > 0$, form a modular operad of projective smooth Deligne-Mumford stacks, $\overline{\mathcal{M}}$ ([GeK98]). Therefore, the homologies of these operads, $H_*(\overline{\mathcal{M}}_0; \mathbb{Q})$ and $H_*(\overline{\mathcal{M}}; \mathbb{Q})$, are cyclic and modular operads respectively.

An important result in the algebraic theory of the Gromov-Witten invariants is that, if X is a complex projective manifold and $\Lambda(X)$ is the Novikov ring of X , the cohomology $H^*(X; \Lambda(X))$ has a natural structure of an algebra over the modular operad $H_*(\overline{\mathcal{M}}; \mathbb{Q})$, and so it is a complete cohomological field theory ([Be], see [Man]).

But there is another modular operad associated to the geometric operad $\overline{\mathcal{M}}$: the modular operad $C_*(\overline{\mathcal{M}}; \mathbb{Q})$ of singular chains. Algebras over this operad have been studied in [GeK98], [KSV] and [KVZ].

In this paper we prove that the modular operad $C_*(\overline{\mathcal{M}}; \mathbb{Q})$ is formal; so it is weakly equivalent to the modular operad of its homology $H_*(\overline{\mathcal{M}}; \mathbb{Q})$. As a consequence, the “up to homotopy” algebras of these two operads are the same.

A paradigmatic example of operad is the little 2-disc operad of Boardman-Vogt, $\mathcal{D}_2(l)$, of configurations of l disjoint discs in the unity disc of \mathbb{R}^2 . Our result can be seen as the analogue for $\overline{\mathcal{M}}$ of the Kontsevich-Tamarkin's formality theorem of for $C_*(\mathcal{D}_2; \mathbb{Q})$ ([Ko] and [T]; moreover [Ko] also explains the relation between this formality theorem, Deligne's conjecture in Hochschild cohomology and Kontsevich's formality theorem in deformation quantization).

Our paper is organized as follows. In section 2, we study symmetric monoidal functors between symmetric monoidal categories, since they induce functors between the categories of their operads. After recalling some definitions and fixing some notations of operads and monoidal categories, we prove a symmetric De Rham theorem. We then introduce the notion of formal symmetric monoidal functor, and we see how this kind of functor produces formal operads.

In section 3, as a consequence of Hodge theory, we prove that the cubic chain functor on the category of compact Kähler manifolds $C_* : \mathbf{K\ddot{a}h} \longrightarrow \mathbf{C}_*(\mathbb{R})$ is a formal symmetric monoidal functor. It follows that, if X is an operad of compact Kähler manifolds, then the operad of chains $C_*(X; \mathbb{R})$ is formal. This is the analogue in the theory of operads of the Deligne-Griffiths-Morgan-Sullivan formality theorem in rational homotopy theory ([DGMS]).

The goal of sections 4, 5 and 6 is to prove the descent of formality from \mathbb{R} to \mathbb{Q} . In section 4 we recall some results due to M. Markl on minimal models of operads in the form that we will use in order to generalize them to cyclic and modular operads.

In section 5, drawing on Deligne's weight theory for Frobenius endomorphism in étale cohomology, we introduce weights and show the formality of the category of complexes endowed with a pure endomorphism. Next, in th. 5.2.4, we prove a characterization of formality of an operad in terms of the lifting of automorphisms of the homology of the operad to automorphisms of the operad itself.

The automorphism group of a minimal operad with homology of finite type is a pro-algebraic group. This result allows us to use the descent theory of algebraic groups to prove the independence of formality of the ground field in th. 6.2.1.

In section 7 we show how the above results can be extended easily to cyclic operads. In particular we obtain the formality of the cyclic operad $C_*(\overline{\mathcal{M}}_0; \mathbb{Q})$.

In the last section, we go one step further and prove the above results also for modular operads. In particular, we introduce minimal models of modular operads and we prove their existence and lifting properties. Here, we follow Grothendieck's idea in his "jeu de Léo-Teichmüller" ([Gro]), in which he builds the complete Teichmüller tower inductively on the modular dimension. Once this is established, the proofs of the previous sections can be transferred to the modular context without difficulty. Finally, we conclude the formality of the modular operad $C_*(\overline{\mathcal{M}}; \mathbb{Q})$.

2. FORMAL OPERADS

2.1. Operads. Let us recall some definitions and notations about operads (see [GiK], [KM], [MSS]).

2.1.1. Let Σ be the *symmetric groupoid*, that is, the category whose objects are the sets $\{1, \dots, n\}$, $n \geq 1$, and the only morphisms are those of the symmetric groups $\Sigma_n = \text{Aut}\{1, \dots, n\}$.

2.1.2. Let \mathcal{C} be a category. The category of contravariant functors from Σ to \mathcal{C} is called the category of Σ -modules and is denoted by $\Sigma\mathbf{Mod}_{\mathcal{C}}$, or just $\Sigma\mathbf{Mod}$ if \mathcal{C} is understood. We identify its objects with sequences of objects in \mathcal{C} , $E = ((E(l)))_{l \geq 1}$, with a right Σ_l -action on each $E(l)$. If e is an element of $E(l)$, l is called the *arity* of e . If E and F are Σ -modules, a *morphism of Σ -modules* $f : E \rightarrow F$ is a sequence of Σ_l -equivariant morphisms $f(l) : E(l) \rightarrow F(l)$, $l \geq 1$.

2.1.3. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. A *unital Σ -operad* (an *operad* for short) in \mathcal{C} is a Σ -module P together with a family of structure morphisms: composition $\gamma_{l; m_1, \dots, m_l} : P(l) \otimes P(m_1) \otimes \dots \otimes P(m_l) \rightarrow P(m_1 + \dots + m_l)$, and unit $\eta : \mathbf{1} \rightarrow P(1)$, satisfying the axioms of equivariance, associativity, and unit. A *morphism of operads* is a morphism of Σ -modules compatible with structure morphisms. Let us denote by $\mathbf{Op}_{\mathcal{C}}$, or simply \mathbf{Op} when \mathcal{C} is understood, the category of operads in \mathcal{C} and its morphisms.

2.2. Symmetric monoidal categories and functors. In the study of Σ -operads the commutativity constraint plays an important role. In particular the functors we are interested in are functors between symmetric monoidal categories which are compatible with the associativity, commutativity and unit constraints.

2.2.1. The following are some of the symmetric monoidal categories we will deal with in this paper. On the one hand, the geometric ones:

Top: the category of topological spaces.

Dif: the category of differentiable manifolds.

Käh: the category of compact Kähler manifolds.

V(\mathbb{C}): the category of smooth projective \mathbb{C} -schemes.

On the other hand, the algebraic categories, which will be subcategories, or variants of

$\mathbf{C}_*(\mathcal{A})$: the category of complexes with a differential of degree -1 of an abelian monoidal symmetric category $(\mathcal{A}, \otimes, \mathbf{1})$. The morphisms are called chain maps. If \mathcal{A} is the category of R -modules for some ring R , we will denote it by $\mathbf{C}_*(R)$. Operads in $\mathbf{C}_*(\mathcal{A})$ are also called *dg operads*.

In a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ we usually denote the natural commutativity isomorphism by $\tau_{X,Y} : X \otimes Y \rightarrow Y \otimes X$. For example, in $\mathbf{C}_*(\mathcal{A})$, the natural commutativity isomorphism

$$\tau_{X,Y} : X \otimes Y \rightarrow Y \otimes X$$

includes the signs:

$$\tau_{X,Y}(x \otimes y) = (-1)^{\deg(x) \deg(y)} y \otimes x .$$

2.2.2. As usual, we move from a geometric category to an algebraic one through a functor. Let us recall (see [KS]) that a *monoidal functor*

$$(F, \kappa, \eta) : (\mathcal{C}, \otimes, \mathbf{1}) \longrightarrow (\mathcal{D}, \otimes, \mathbf{1}')$$

between monoidal categories is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ together with a natural morphism of \mathcal{D} ,

$$\kappa_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y),$$

for all objects $X, Y \in \mathcal{C}$, and a morphism of \mathcal{D} , $\eta : \mathbf{1}' \longrightarrow F\mathbf{1}$, compatibles with the constraints of associativity, and unit. We will refer the *Künneth morphism* as κ .

If \mathcal{C} and \mathcal{D} are symmetric monoidal categories, a monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be *symmetric* if κ is compatible with the commutativity constraint.

For example, the *homology functor* $H_* : \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A})$ is a symmetric monoidal functor, taking the usual Künneth morphism

$$H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \otimes Y).$$

as κ .

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two monoidal functors. A natural transformation $\phi : F \Rightarrow G$ is said to be *monoidal* if it is compatible with κ and η .

2.2.3. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a symmetric monoidal functor. It is easy to prove that, applied componentwise, F induces a functor between Σ -operads

$$\mathbf{Op}_F : \mathbf{Op}_{\mathcal{C}} \longrightarrow \mathbf{Op}_{\mathcal{D}},$$

also denoted by F .

In particular, for an operad $P \in \mathbf{Op}_{\mathbf{C}_*(\mathcal{A})}$, its homology is an operad $HP \in \mathbf{Op}_{\mathbf{C}_*(\mathcal{A})}$.

In the same way, if $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ are two symmetric monoidal functors, a monoidal natural transformation $\phi : F \Rightarrow G$ induces a natural transformation

$$\mathbf{Op}_{\phi} : \mathbf{Op}_F \Rightarrow \mathbf{Op}_G,$$

also denoted by ϕ .

2.3. Weak equivalences. We will use weak equivalences in several contexts.

2.3.1. Let X and Y be objects of $\mathbf{C}_*(\mathcal{A})$. A chain map $f : X \longrightarrow Y$ is said to be a *weak equivalence of complexes* if the induced morphism $f_* = Hf : HX \longrightarrow HY$ is an isomorphism.

2.3.2. Let \mathcal{C} be a category, \mathcal{A} an abelian category, and $F, G : \mathcal{C} \rightrightarrows \mathbf{C}_*(\mathcal{A})$ two functors. A natural transformation $\phi : F \Rightarrow G$ is said to be a *weak equivalence of functors*, if the morphism $\phi(X) : F(X) \rightarrow G(X)$ is a weak equivalence, for every object X in \mathcal{C} .

2.3.3. A morphism $\rho : P \longrightarrow Q$ of operads in $\mathbf{C}_*(\mathcal{A})$ is said to be a *weak equivalence of operads* if $\rho(l) : P(l) \longrightarrow Q(l)$ is a weak equivalence of chain complexes, for all l .

2.3.4. Let \mathcal{C} be a category endowed with a distinguished class of morphism called *weak equivalences*. We suppose that this is a saturated class of morphisms which contains all isomorphisms. Two objects X and Y of \mathcal{C} are said to be *weakly equivalent* if there exists a sequence of morphism of \mathcal{C}

$$X \longleftarrow X_1 \longrightarrow \cdots \longleftarrow X_{n-1} \longrightarrow Y.$$

which are weak equivalences. If X and Y are weakly equivalent, we say that Y is a *model* of X .

2.3.5. The following proposition is an easy consequence of the definitions.

Proposition 2.3.1. *If $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ are two weakly equivalent symmetric monoidal functors, the functors \mathbf{Op}_F and \mathbf{Op}_G are weakly equivalent. In particular, for every operad P in \mathcal{C} , the operads $F(P)$ and $G(P)$ are weakly equivalent.*

2.4. **Symmetric De Rham's theorem.** In this section we will demonstrate a symmetric version of De Rham's theorem comparing the complex of singular cubic chains and De Rham's complex of currents, including its symmetric structure as monoidal functors.

2.4.1. It is well known that the functor of *singular chains*

$$S_*(\ ; \mathbb{Z}) : \mathbf{Top} \longrightarrow \mathbf{C}_*(\mathbb{Z}) ,$$

together with the *shuffle product* as the Künneth morphism, is a symmetric monoidal functor.

On the other hand, let $\mathcal{D}'_*(M)$ be the complex of De Rham's *currents* of a differentiable manifold M ; that is, $\mathcal{D}'_*(M)$ is the topological dual of the complex $\mathcal{D}^*(M)$ of differential forms with compact support. Then the functor

$$\mathcal{D}'_* : \mathbf{Dif} \longrightarrow \mathbf{C}_*(\mathbb{R}) ,$$

is a symmetric monoidal functor with the Künneth morphism

$$\kappa_{M,N} : \mathcal{D}'_*(M) \otimes \mathcal{D}'_*(N) \longrightarrow \mathcal{D}'_*(M \times N)$$

induced by the tensor product of currents. Thereby, if $S \in \mathcal{D}'_*(M)$, and $T \in \mathcal{D}'_*(N)$, then

$$\langle \kappa(S \otimes T) , \pi_M^*(\omega) \wedge \pi_N^*(\nu) \rangle = \langle S , \omega \rangle \cdot \langle T , \nu \rangle$$

for all $\omega \in \mathcal{D}^*(M)$ and $\nu \in \mathcal{D}^*(N)$.

In order to compare the functor of currents with the functor of singular chains on differentiable manifolds, one can consider the complex of chains $S_*^\infty(M)$ generated by the \mathcal{C}^∞ -maps $\Delta^p \longrightarrow M$. The corresponding functor of \mathcal{C}^∞ -singular chains $S_*^\infty : \mathbf{Dif} \longrightarrow \mathbf{C}_*(\mathbb{Z})$ is also a symmetric monoidal functor with the shuffle product.

On the one hand, the natural inclusion of \mathcal{C}^∞ -singular chains into singular ones defines a monoidal natural transformation $S_*^\infty \Rightarrow S_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{Z})$, and from the approximation theorem it follows that it is a weak equivalence of functors. On the other hand, by Stokes' theorem, integration along \mathcal{C}^∞ -singular simplexes induces a natural transformation $\int : S_*^\infty \Rightarrow \mathcal{D}'_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{R})$, which is a weak equivalence of functors by De Rham's theorem.

So the functors of singular chains and currents are weakly equivalent. However, the natural transformation \int is not compatible with the monoidal structures. To overcome this deficiency, we will use the cubic singular chains instead of the simplicial ones.

2.4.2. For a topological space X , cubic chains are generated by the singular cubes of X ; that is, the continuous maps $I^p \rightarrow X$, where I is the unit interval of the real line \mathbb{R} and $p \in \mathbb{N}$, modulo the degenerate ones (see e.g. [Mas]). The functor of *cubic chains*

$$C_*(\ ; \mathbb{Z}) : \mathbf{Top} \rightarrow \mathbf{C}_*(\mathbb{Z}) ,$$

with the *cross product*

$$\times : C_*(X; \mathbb{Z}) \otimes C_*(Y; \mathbb{Z}) \rightarrow C_*(X \times Y; \mathbb{Z}) ,$$

which for singular cubes $c : I^p \rightarrow X$ and $d : I^q \rightarrow Y$ is defined as the cartesian product

$$c \times d : I^{p+q} = I^p \times I^q \rightarrow X \times Y ,$$

is a monoidal functor.

If M is a differentiable manifold, one may consider chains generated by \mathcal{C}^∞ -maps $I^p \rightarrow M$. The corresponding functor of \mathcal{C}^∞ -*cubic chains* $C_*^\infty(\ ; \mathbb{Z}) : \mathbf{Dif} \rightarrow \mathbf{C}_*(\mathbb{Z})$ is also a monoidal functor.

Finally the inclusion $C_*^\infty \Rightarrow C_* : \mathbf{Dif} \rightarrow \mathbf{C}_*(\mathbb{Z})$ and integration $\int : C_*^\infty \Rightarrow \mathcal{D}'_* : \mathbf{Dif} \rightarrow \mathbf{C}_*(\mathbb{R})$ are monoidal natural transformations, which are weak equivalences of monoidal functors.

In spite of this, in this case another problem arises: the monoidal functor of cubic chains is not symmetric, because the cross product is not commutative.

2.4.3. However, over \mathbb{Q} , or more generally over a \mathbb{Q} -algebra, we can symmetrize cross product with the classical alternating operator. For every n , define the map $\text{Alt} : C_n(X; \mathbb{Q}) \rightarrow C_n(X; \mathbb{Q})$ as

$$\text{Alt}(c) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} c \circ \sigma ,$$

for all singular cubes $c : I^p \rightarrow X$. The map Alt is well defined because, if c is a degenerate singular cube, then $\text{Alt}(c)$ is a degenerate cubic chain.

Proposition 2.4.1. *The functor of cubic chains $C_*(\ ; \mathbb{Q}) : \mathbf{Top} \rightarrow \mathbf{C}_*(\mathbb{Q})$, together with the product*

$$\kappa_{X,Y} : C_*(X; \mathbb{Q}) \otimes C_*(Y; \mathbb{Q}) \rightarrow C_*(X \times Y; \mathbb{Q}) ,$$

defined by $\kappa(c \otimes d) = \text{Alt}(c \times d)$, is a symmetric monoidal functor.

Proof. First of all, compatibility with units is trivial. Next, compatibility with associativity and commutativity constraints is proved in a similar way to the usual proofs of the associativity and commutativity properties for the wedge product of multi-linear alternating tensors. Finally, the fact that Alt is a chain map is possibly a classical result, but we have not found a proof in the literature. So let us see that Alt is a chain map. Recall the definition of the differential $d : C_n(X; \mathbb{Q}) \rightarrow C_{n-1}(X; \mathbb{Q})$ of the cubic chain complex. For $1 \leq i \leq n$, $\epsilon \in \{0, 1\}$, let

$\delta_i^\epsilon : I^{n-1} \longrightarrow I^n$ denote the face defined by $\delta_i^\epsilon(t_1, \dots, t_{n-1}) = (t_1, \dots, \epsilon, \dots, t_{n-1})$, where ϵ is in the i -th place. Now, if $c \in C_n(X, \mathbb{Q})$, $d(c)$ is defined by

$$d(c) = \sum_{i, \epsilon} (-1)^{i+\epsilon} c \circ \delta_i^\epsilon,$$

and we have

$$d(\text{Alt}(c)) = \frac{1}{n!} \sum_{i, \epsilon} \sum_{\sigma \in \Sigma_n} (-1)^{i+\epsilon+|\sigma|} c \circ \sigma \circ \delta_i^\epsilon,$$

and

$$\text{Alt}(dc) = \frac{1}{(n-1)!} \sum_{\tau \in \Sigma_{n-1}} \sum_{r, \epsilon} (-1)^{|\tau|+r+\epsilon} c \circ \delta_r^\epsilon \circ \tau$$

Then, it is easy to check the following claim

Claim: For all $\tau \in \Sigma_{n-1}$, and $r, i \in \{1, \dots, n\}$ let

$$\sigma_{\tau, r, i} := (r, \dots, n) \circ \bar{\tau} \circ (i, \dots, n)^{-1},$$

where $\bar{\tau} \in \Sigma_n$ is defined by $\bar{\tau}(i) = \tau(i)$ for all $1 \leq i < n$, and (r, \dots, n) denotes the cycle

$$(1, 2, \dots, r-1, r, r+1, \dots, n) \mapsto (1, 2, \dots, r-1, r+1, \dots, n, r).$$

Then $\sigma_{\tau, r, i}$ is the only permutation that satisfies $\sigma_{\tau, r, i} \circ \delta_i^\epsilon = \delta_r^\epsilon \circ \tau$.

Moreover, the set map

$$\begin{aligned} \Sigma_{n-1} \times \{1, \dots, n\} \times \{1, \dots, n\} &\longrightarrow \Sigma_n \times \{1, \dots, n\} \\ (\tau, r, i) &\mapsto (\sigma_{\tau, r, i}, i) \end{aligned}$$

is bijective.

To finish checking that $d \circ \text{Alt} = \text{Alt} \circ d$, it suffices to note that, from $(-1)^{|\sigma_{\tau, r, i}|+i} = (-1)^{|\tau|+r}$, it follows

$$\frac{1}{n} \sum_{i=1}^n (-1)^{|\sigma|+i+\epsilon} c \circ \sigma_{\tau, r, i} \circ \delta_i^\epsilon = (-1)^{|\tau|+r+\epsilon} c \circ \delta_r^\epsilon \circ \tau$$

for all ϵ, r, τ . □

The functor of \mathcal{C}^∞ -cubic chains $C_*^\infty(-; \mathbb{Q}) : \mathbf{Dif} \longrightarrow \mathbf{C}_*(\mathbb{Q})$ is also a symmetric monoidal functor with the corresponding product.

We will now check that integration is compatible with the monoidal structure.

Lemma 2.4.2. *Integration along \mathcal{C}^∞ -cubes induces a monoidal natural transformation*

$$\int : C_*^\infty \Rightarrow \mathcal{D}'_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{R}),$$

which is a weak equivalence of symmetric monoidal functors.

Proof. Let M be a differentiable manifold and $c : I^p \longrightarrow M$ a \mathcal{C}^∞ -singular cube. Integration along c , $\int_c^M \omega = \int_{I^p} c^*(\omega)$, defines a chain map

$$\int^M : C_*^\infty(M; \mathbb{R}) \longrightarrow \mathcal{D}'_*(M; \mathbb{R})$$

by $c \mapsto \int_c^M$, which is obviously natural in M , and a weak equivalence by De Rham's theorem.

The natural transformation \int^M is monoidal, by Fubini's theorem and by the change of variables theorems. Indeed, for $c : I^p \longrightarrow M$ and $d : I^q \longrightarrow N$ \mathcal{C}^∞ -singular cubes, and $\omega \in \mathcal{D}^p(M)$ and $\nu \in \mathcal{D}^q(N)$, we have

$$\begin{aligned} \int_{\kappa(c \otimes d)}^{M \times N} \pi_M^*(\omega) \wedge \pi_N^*(\nu) &= \frac{1}{(p+q)!} \sum_{\sigma \in \Sigma_{p+q}} (-1)^{|\sigma|} \int_{(c \times d) \circ \sigma}^{M \times N} \pi_M^*(\omega) \wedge \pi_N^*(\nu) \\ &= \int_{c \times d}^{M \times N} \pi_M^*(\omega) \wedge \pi_N^*(\nu) \\ &= \left(\int_c^M \pi_M^*(\omega) \right) \left(\int_d^N \pi_N^*(\nu) \right) \\ &= \left(\kappa \circ \left(\int_c^M \otimes \int_d^N \right) \right) (\pi_M^*(\omega) \wedge \pi_N^*(\nu)) . \end{aligned}$$

□

To sum up, we can state the following symmetric version of De Rham's theorem

Theorem 2.4.3. *The functors of cubic chains and currents*

$$C_*, \mathcal{D}'_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{R})$$

are weakly equivalent symmetric monoidal functors.

Remark 2.4.4. A similar result can be obtained with the functor of *oriented cubic chains* C_*^{or} , used by Kontsevich (see [Ko], (2.2)), which, with the cross product, is a monoidal symmetric functor from \mathbf{Top} to $\mathbf{C}_*(\mathbb{Z})$. It can be proved using the alternating operator that, at least over \mathbb{Q} , computes the usual homology of a topological space. This solution is equivalent to the previous one, because the natural projection $C_*(\ ; \mathbb{Q}) \longrightarrow C_*^{\text{or}}(\ ; \mathbb{Q})$ is a monoidal natural transformation, which is a weak equivalence of symmetric monoidal functors.

Remark 2.4.5. The above results can be seen as analogous in the covariant setting to the following well known statements. The contravariant functor of singular cochains $S^* : \mathbf{Top}^{op} \longrightarrow \mathbf{C}^*(\mathbb{Z})$, together with the Alexander-Whitney product, is a monoidal contravariant functor. Although the functor of \mathcal{C}^∞ -singular cochains $S_\infty^* : \mathbf{Dif}^{op} \longrightarrow \mathbf{C}^*(\mathbb{R})$ is weakly equivalent to the symmetric monoidal functor of differential forms $\mathcal{E}^* : \mathbf{Dif}^{op} \longrightarrow \mathbf{C}^*(\mathbb{R})$, the monoidal functor S_∞^* is symmetric only up to homotopy. \mathcal{E}^* can be topologically defined as a symmetric monoidal functor using the Sullivan's \mathbb{Q} -cdga $Su_{\mathbb{Q}}$ of simplicial differential forms, which defines a symmetric monoidal contravariant functor $\mathbf{Top}^{op} \longrightarrow \mathbf{C}^*(\mathbb{Q})$, together with a weak equivalence of symmetric monoidal functors $\mathcal{E}^* \Rightarrow Su_{\mathbb{R}} : \mathbf{Dif} \longrightarrow \mathbf{C}^*(\mathbb{R})$.

2.5. Formality.

2.5.1. The notion of formality has attracted interest since Sullivan's work on rational homotopy theory. In the operadic setting the notion of formality appears in [Mkl] and [Ko].

Definition 2.5.1. An operad P in $\mathbf{C}_*(\mathcal{A})$ is said to be *formal* if it is weakly equivalent to its homology HP .

More generally, we can give the following definition

Definition 2.5.2. Let \mathcal{C} be a category endowed with an idempotent endofunctor $H : \mathcal{C} \longrightarrow \mathcal{C}$, and take as weak equivalences the morphisms $f : X \longrightarrow Y$ such that $H(f)$ is an isomorphism. An object X of \mathcal{C} is said to be *formal* if X and HX are weakly equivalent.

2.5.2. In particular, a functor $F : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathcal{A})$ is *formal* if it is weakly equivalent to its homology $HF : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathcal{A})$. However, we will use this notion in the context of symmetric monoidal functors. So, the definition of formality in this case is

Definition 2.5.3. Let \mathcal{C} be a symmetric monoidal category, and $F : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathcal{A})$ a symmetric monoidal functor. It is said that F is a *formal symmetric monoidal functor* if F and HF are weakly equivalent in the category of symmetric monoidal functors.

If the identity functor of $\mathbf{C}_*(\mathcal{A})$ is a formal symmetric functor, $\mathbf{C}_*(\mathcal{A})$ is said to be a *formal symmetric monoidal category*.

Let \mathcal{B} be a symmetric monoidal subcategory of $\mathbf{C}_*(\mathcal{A})$. If the inclusion functor $\mathcal{B} \longrightarrow \mathbf{C}_*(\mathcal{A})$ is a formal symmetric monoidal functor, we will say that \mathcal{B} is a *formal symmetric monoidal subcategory* of $\mathbf{C}_*(\mathcal{A})$.

The properties below follow immediately from the definitions.

Proposition 2.5.4. *Let \mathcal{B} be a formal symmetric monoidal subcategory of $\mathbf{C}_*(\mathcal{A})$. If $F : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathcal{A})$ is a monoidal functor with values in the subcategory \mathcal{B} , then F is a formal symmetric monoidal functor.*

Proposition 2.5.5. *Let $F : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathcal{A})$ be a functor. If F is a formal symmetric monoidal functor, then*

$$F : \mathbf{Op}_{\mathcal{C}} \longrightarrow \mathbf{Op}_{\mathbf{C}_*(\mathcal{A})}$$

sends operads in \mathcal{C} to formal operads in $\mathbf{C}_(\mathcal{A})$.*

2.5.3. Let R be a commutative ring, and $R - \mathbf{cdga}$ the category of differential graded-commutative R -algebras, or simply \mathbf{cdg} R -algebras. It is a symmetric monoidal category. Then, if \mathcal{C} a symmetric monoidal category and $F : \mathcal{C} \longrightarrow \mathbf{C}^*(R)$ is a symmetric monoidal functor, it is a well known fact that F induces a functor from the category of commutative monoids of $(\mathcal{C}, \otimes, \mathbf{1})$ to \mathbf{cdg} R -algebras.

Besides, if F is a formal symmetric monoidal functor, then F sends commutative monoids to formal \mathbf{cdg} R -algebras.

If \mathcal{C} is a category with finite products, and a final object $\mathbf{1}$, then $(\mathcal{C}, \times, \mathbf{1})$, is a symmetric monoidal category. In this case every object X of \mathcal{C} is a comonoid object with the diagonal $X \rightarrow X \times X$ and the unit $X \rightarrow \mathbf{1}$. So we have

Proposition 2.5.6. *Let \mathcal{C} be a category with finite products, and a final object $\mathbf{1}$. Every formal symmetric monoidal contravariant functor $F : \mathcal{C}^{op} \rightarrow \mathbf{C}^*(R)$ sends objects in \mathcal{C} to formal cdg R -algebras.*

3. HODGE THEORY IMPLIES FORMALITY

In [DGMS], Deligne et al. prove the formality of the De Rham cdg algebra of a compact Kähler manifold. In this section we will see how this result can be mimicked for the cubic chain complex of an operad of compact Kähler manifolds.

3.1. Formality of De Rham's functor. In [DGMS], the first of the proofs of formality th. 5.22 relies on the *Hodge decomposition* for the complex of forms and the Kähler identities. From them, the *dd^c-lemma* is proved and the existence of a diagram of complexes, called *d^c-diagram*,

$$(\mathcal{E}^*(M), d) \longleftarrow ({}^c\mathcal{E}^*(M), d) \longrightarrow (H_{d^c}^*(M), d) ,$$

is deduced. Here M is a compact Kähler manifold, $\mathcal{E}^*(M)$ is the real De Rham complex of M , ${}^c\mathcal{E}^*(M)$ the subcomplex of d^c -closed forms, and $H_{d^c}^*(M)$ the quotient complex ${}^c\mathcal{E}^*(M)/d^c(\mathcal{E}^*(M))$. In the *d^c-diagram*, both maps are weak equivalences of chain complexes, and the differential induced by d on $H_{d^c}^*(M)$ is zero. Since ${}^c\mathcal{E}^*$ is a symmetric monoidal subfunctor of \mathcal{E}^* and the morphisms of this diagram are natural, the functor \mathcal{E}^* is formal. So the theorem of formality can also be stated with the previous definitions as follows

Theorem 3.1.1. *The functor of differential forms $\mathcal{E}^* : \mathbf{Käh}^{op} \rightarrow \mathbf{C}^*(\mathbb{R})$ is a formal symmetric monoidal functor.*

This result, together with prop. 2.5.6, implies the formality theorem for De Rham's cdg-algebra in its usual formulation: The De Rham functor $\mathcal{E}^* : \mathbf{Käh}^{op} \rightarrow \mathbb{R} - \mathbf{cdga}$ sends objects in $\mathbf{Käh}$ to formal cdg \mathbb{R} -algebras.

3.2. Formality of the current complex functor. We claim that an analogous theorem of formality is obtained replacing forms with currents.

Theorem 3.2.1. *The functor of currents $\mathcal{D}'_* : \mathbf{Käh} \rightarrow \mathbf{C}_*(\mathbb{R})$ is a formal symmetric monoidal functor.*

Proof. Let M be a compact Kähler manifold. It is a classical result of Hodge theory (see [Sch]) that the Kähler identities between the operators d, d^c, Δ, \dots of the De Rham complex of differential forms are also satisfied by the corresponding dual operators on De Rham complex of currents. Hence we have the following *dd^c-lemma*.

Lemma 3.2.2. *Let T be a d^c -closed and d -exact current. Then, there exists a current S such that $T = dd^c S$.*

From this lemma, we can follow verbatim the first proof of theorem 5.22 (*the d^c -Diagram Method*) in [DGMS] and we obtain a d^c -diagram for currents:

$$(\mathcal{D}'_*(M), d) \longleftarrow ({}^c\mathcal{D}'_*(M), d) \longrightarrow (H_*^{d^c}(M), d) .$$

Here ${}^c\mathcal{D}'_*(M)$ denotes the subcomplex of $\mathcal{D}'_*(M)$ defined by the d^c -closed currents, and $H_*^{d^c}(M)$ is the quotient ${}^c\mathcal{D}'_*(M)/d^c({}^c\mathcal{D}'_*(M))$. In this d^c -diagram both maps are weak equivalences, and the differential induced by d on the latter is zero. So we have $H_*^{d^c}(M) \cong H_*(\mathcal{D}'_*(M))$.

Now, since d^c satisfies the Leibnitz rule, ${}^c\mathcal{D}'_*$ is a symmetric monoidal subfunctor of \mathcal{D}'_* .

Finally, since the morphisms of the above d^c -diagram are natural and compatible with the Künneth morphism, it follows that \mathcal{D}'_* is a formal symmetric monoidal functor. \square

As a consequence of the formality of the current functor and the symmetric De Rham theorem for currents (th. 2.4.3), the formality of the cubic chains functor for compact Kähler manifolds follows.

Corollary 3.2.3. *The functor of cubic chains $C_*(\cdot; \mathbb{R}) : \mathbf{Käh} \longrightarrow \mathbf{C}_*(\mathbb{R})$ is a formal symmetric monoidal functor.*

3.3. Formality of Kählerian operads. From 2.5.5 and 3.2.3 we obtain the operadic version of the formality DGMS theorem.

Theorem 3.3.1. *If X is an operad in $\mathbf{Käh}$, then the operad of cubic chains $C_*(X; \mathbb{R})$ is formal.*

This result, together with prop. 2.5.5 and the descent theorem th. 6.2.1 below, implies the formality of the operad of cubic chains with rational coefficients for every operad of compact Kähler manifolds (see cor. 6.3.1 below).

3.4. Formality of DM-operads. The above results can be easily generalized to the category of *Deligne-Mumford projective and smooth stacks* over \mathbb{C} , which we will denote by $\mathbf{DMV}(\mathbb{C})$.

Indeed, every stack of this kind defines a compact Kähler V -manifold and for such V -manifolds we have the functors of cubic chains, \mathcal{C}^∞ -cubic chains and currents, and also Hodge theory (see [Ba]). This allows us to obtain an analogous result to cor. 3.2.3:

Theorem 3.4.1. *The functor of cubic chains $C_*(\cdot; \mathbb{R}) : \mathbf{DMV}(\mathbb{C}) \longrightarrow \mathbf{C}_*(\mathbb{R})$ is a formal symmetric monoidal functor.*

And, from 2.5.5 follows

Theorem 3.4.2. *If X is an operad in $\mathbf{DMV}(\mathbb{C})$, then the operad of cubic chains $C_*(X; \mathbb{R})$ is formal.*

4. MINIMAL OPERADS

In this section \mathbf{k} will denote a field of characteristic zero, and an operad will be an operad in the category of dg vector spaces over \mathbf{k} , $\mathbf{C}_*(\mathbf{k})$. The category of these operads is denoted simply by \mathbf{Op} . It is a complete and cocomplete category (see [Hi]).

4.1. Some preliminaries. Let us start by recalling some basic results on minimal operads due to M. Markl ([Mkl], see [MSS]).

4.1.1. A *minimal operad* is an operad of the form $(\Gamma(V), d_M)$, where $\Gamma : \Sigma\mathbf{Mod} \longrightarrow \mathbf{Op}$ is the free operad functor, V is a Σ -module with zero differential with $V(1) = 0$, and the differential d_M is *decomposable*.

The free operad functor $\Gamma : \Sigma\mathbf{Mod} \longrightarrow \mathbf{Op}$ is a right adjoint functor for the forgetful functor $U : \mathbf{Op} \longrightarrow \Sigma\mathbf{Mod}$.

A *minimal model* of an operad P is a minimal operad P_∞ , together with a weak equivalence $P_\infty \longrightarrow P$.

Let $P = (P(l))_{l \geq 1}$ be an operad. M. Markl has proved that, if $HP(1) = \mathbf{k}$, P has a minimal model P_∞ with $P_\infty(1) = \mathbf{k}$ ([MSS], th. 3.125).

As observed in [MSS], remark II.1.62, the category of operads P with $P(1) = \mathbf{k}$ is equivalent to the category of *pseudo-operads* Q with $Q(1) = \mathbf{0}$, the zero dg vector space (see *op. cit.* def. II.1.16).

In the sequel, we will work only with pseudo-operads, with $HP(1) = 0$, and we will call them simply operads. We will denote by \mathbf{Op} the category of these operads, and

$$\circ_i : P(l) \otimes P(m) \longrightarrow P(l + m - 1), \quad 1 \leq i \leq l,$$

their composition operations.

4.2. Truncated operads. We will now introduce the arity truncation and their right and left adjoints, which enables us to introduce in the operadic setting the analogs of the skeleton and coskeleton functors of simplicial set theory.

Here we establish the results for the arity truncation in a form that can be easily translated to modular operads in §8.

4.2.1. Let $E = (E(l))_{l \geq 1}$ be a Σ -module, and $n \geq 1$ an integer. The grading of E induces a decreasing filtration $(E(\geq l))_{l \geq 1}$, by the sub- Σ -modules

$$E(\geq l) := (E(i))_{i \geq l}.$$

Let P be an operad. We will denote by $P \cdot P(n)$ the sub- Σ -module consisting of elements $\alpha \circ_i \beta$ with $\alpha \in P(l)$ and $\beta \in P(m)$, such that at least one of l, m is n . It follows from the definitions that

$$P \cdot P(n) \subset P(\geq n).$$

If, moreover, $P(1) = 0$, then

$$P \cdot P(n) \subset P(\geq n+1).$$

The first property implies that $P(\geq n)$ is an ideal of P , so the quotient $P/P(\geq n+1)$ is an operad, which is zero in arities $> n$. This is a so-called n -truncated operad. However, we find it more natural to give the following definition of n -truncated operad.

Definition 4.2.1. A n -truncated operad is a finite sequence of objects in $\mathbf{C}(\mathbf{k})$,

$$P = (P(1), \dots, P(n)),$$

with a right Σ_l -action on each $P(l)$, together with a family of composition operations, satisfying those axioms of composition operations in \mathbf{Op} that make sense for truncated operads. A *morphism of n -truncated operads* $f : P \longrightarrow Q$ is a finite sequence of morphisms of Σ_l -modules $f(l) : P(l) \longrightarrow Q(l)$, $1 \leq l \leq n$, which commute with composition operations.

Let $\mathbf{Op}(\leq n)$ denote the category of n -truncated operads of $\mathbf{C}_*(\mathbf{k})$.

A *weak equivalence* of n -truncated operads is a morphism of n -truncated operads $\phi : P \longrightarrow Q$ which induces isomorphisms of graded \mathbf{k} -vector spaces, $H\phi(l) : HP(l) \longrightarrow HQ(l)$, for $l = 1, \dots, n$.

Given an operad P , $t_n P := (P(1), \dots, P(n))$ defines a *truncation functor*

$$t_n : \mathbf{Op} \longrightarrow \mathbf{Op}(\leq n) .$$

4.2.2. For a n -truncated operad P denote by $t_* P$ the Σ -module that is $\mathbf{0}$ in arities $> n$ and coincides with P in arities $\leq n$. Since $P \cdot P(n) \subset P(\geq n)$, $t_* P$ together with the structural morphisms of P trivially extended, is an operad, and the proposition below follows easily from the definitions

Proposition 4.2.2. *Let $n \geq 1$ be an integer. Then*

- (1) $t_* : \mathbf{Op}(\leq n) \longrightarrow \mathbf{Op}$ is a right adjoint functor for t_n .
- (2) There exists a canonical isomorphism $t_n \circ t_* \cong \text{Id}_{\mathbf{Op}(\leq n)}$.
- (3) t_* is a fully faithful functor.
- (4) t_* preserves limits.
- (5) For $m \geq n$, there exists a natural morphism

$$\psi_{m,n} : t_* t_m \longrightarrow t_* t_n$$

such that $\psi_{l,n} = \psi_{m,n} \circ \psi_{l,m}$, for $l \geq m \geq n$. For an operad P , the family $(t_* t_n P)_n$, with the morphisms $\psi_{m,n}$, is an inverse system of operads. The family of unit morphisms of the adjunctions

$$\psi_n : P \longrightarrow t_* t_n P,$$

induces an isomorphism $\psi : P \longrightarrow \varprojlim t_* t_n P$.

- (6) Let P, Q be operads. If $n \geq 2$, $t_{n-1} P = 0$, and $Q \cong t_* t_n Q$, then

$$\text{Hom}_{\mathbf{Op}}(P, Q) \cong \text{Hom}_{\Sigma_n}(P(n), Q(n)).$$

4.2.3. On the other hand, the functor t_n also has a left adjoint. For a n -truncated operad P , denote by $t_! P$ the operad obtained freely adding to P the operations generated in arities $> n$, that is,

$$t_! P = \Gamma(Ut_* P) / J ,$$

where J is the ideal in $\Gamma(Ut_* P)$ generated by the kernel of $t_n \Gamma(Ut_* P) \longrightarrow P$.

Proposition 4.2.3. *Let $n \geq 1$ be an integer. Then*

- (1) $t_! : \mathbf{Op}(\leq n) \longrightarrow \mathbf{Op}$ is a left adjoint functor for t_n .
- (2) There exists a canonical isomorphism $t_n \circ t_! \cong \text{Id}_{\mathbf{Op}(\leq n)}$.
- (3) $t_!$ is a fully faithful functor.
- (4) $t_!$ preserves colimits.
- (5) For $m \leq n$, there exists a natural morphism

$$\phi_{m,n} : t_! t_m \longrightarrow t_! t_n$$

such that $\phi_{l,n} = \phi_{m,n} \circ \phi_{l,m}$, for $l \leq m \leq n$. For an operad P , the family $(t_! t_n P)_n$, with the morphisms $\phi_{m,n}$, is a directed system of operads. The family of unit morphisms of the adjunctions

$$\phi_n : t_! t_n P \longrightarrow P,$$

induces an isomorphism $\phi : \varinjlim t_! t_n P \longrightarrow P$.

- (6) Let P, Q be operads. If $t_{n-1} Q = 0$, $P \cong t_! t_n P$, and $P(1) = 0$, then

$$\text{Hom}_{\mathbf{Op}}(P, Q) \cong \text{Hom}_{\Sigma_n}(P(n), Q(n)).$$

Proof. Part (1) follows from the definition of $t_!$, and the remaining parts follow from (1) and prop. 4.2.2. \square

We will call the direct system of operads given by

$$0 \rightarrow t_! t_1 P \rightarrow \cdots \rightarrow t_! t_{n-1} P \rightarrow t_! t_n P \rightarrow \cdots$$

the *canonical tower* of P .

As an easy consequence of the existence of right and left adjoint functors for t_n we obtain the following result.

Corollary 4.2.4. *The truncation functors t_n preserve limits and colimits. In particular, they commute with homology, send weak equivalences to weak equivalences, and preserve formality.*

4.3. Principal extensions. Next, we recall the definition of a principal extension of operads and show that the canonical tower of a minimal operad is a sequence of principal extensions. This will allow us to extend these notions to the truncated setting.

4.3.1. To begin with, we establish some notations on suspension and mapping cones of complexes in an additive category.

If A is a chain complex and n is an integer, we denote by $A[n]$ the complex defined by $A[n]_i = A_{i-n}$ with the differential given by $d_{A[n]} = (-1)^n d_A$.

For a chain map $\eta : B \longrightarrow A$ we will denote by $C\eta$, or by $A \oplus_\eta B[1]$, the mapping cone of η , that is to say, the complex that in degree i is given by $(C\eta)_i = A_i \oplus B_{i-1}$ with the differential $d(a, b) = (d_A a + \eta b, -d_B b)$. Therefore $C\eta$ comes with a canonical chain map $i_A : A \longrightarrow C\eta$ and a canonical homogeneous map of graded objects $j_B : B[1] \longrightarrow C\eta$.

For a chain complex X , a chain map $\phi : C\eta \longrightarrow X$ is determined by the chain map $\phi i_A : A \longrightarrow X$ together with the homogeneous map $\phi j_B : B[1] \longrightarrow X$. Conversely, if $f : A \longrightarrow X$ is a

chain map and $g : B[1] \longrightarrow X$ is a homogeneous map such that $f\eta = d_X g + g d_B$, that is, g is a homotopy between $f\eta$ and 0, then there exists a unique chain map $\phi : C\eta \longrightarrow X$ such that $\phi i_A = f$ and $\phi j_B = g$. In other words, $C\eta$ represents the functor $h_\eta : \mathbf{C}_*(\mathbf{k}) \longrightarrow \mathbf{Sets}$ defined, for $X \in \mathbf{C}_*(\mathbf{k})$, by

$$h_\eta(X) = \{(f, g); f \in \text{Hom}_{\mathbf{C}_*(\mathbf{k})}(A, X), g \in \text{Hom}_{\mathbf{k}}(B, X)_1, d_X g + g d_B = f\eta\},$$

where $\text{Hom}_{\mathbf{k}}(B, X)_1$ denotes the set of homogeneous maps of degree 1 of graded \mathbf{k} -vectorial spaces.

4.3.2. Recall the construction of standard cofibrations introduced in ([Hi]). Let P be an operad, V a dg Σ -module and $\xi : V[-1] \longrightarrow P$ a chain map of dg Σ -modules. The standard cofibration associated to these data, denoted by $P\langle V, \xi \rangle$ in [Hi], is an operad that represents the functor $h_\xi : \mathbf{Op} \longrightarrow \mathbf{Sets}$ defined, for $Q \in \mathbf{Op}$, by

$$h_\xi(Q) = \{(f, g); f \in \text{Hom}_{\mathbf{Op}}(P, Q), g \in \text{Hom}_{\mathbf{Gr}\Sigma\text{Mod}}(V, UQ)_0, d_Q g - g d_V = f\xi\},$$

where $\text{Hom}_{\mathbf{Gr}\Sigma\text{Mod}}(V, UQ)_0$ denotes the set of homogeneous maps of degree 0 of graded Σ -modules. When V has zero differential, this construction is called a *principal extension* and denoted by $P *_\xi \Gamma(V)$, see [MSS]. For reasons that will become clear at once, we will denote it by $P \sqcup_\xi V$. From the definition it follows that $P \sqcup_\xi V$ comes with a canonical morphism of operads $i_P : P \longrightarrow P \sqcup_\xi V$ and a canonical homogeneous map of degree 0 of graded Σ -modules $j_V : V \longrightarrow P \sqcup_\xi V$.

Now, one can express $P \sqcup_\xi V$ as a push-out. Let $C(V[-1])$ be the mapping cone of $\text{id}_{V[-1]}$, $S(V) = \Gamma(V[-1])$, $T(V) = \Gamma(C(V[-1]))$, $i_V : S(V) \longrightarrow T(V)$ the morphism of operads induced by the canonical chain map $i : V[-1] \longrightarrow C(V[-1])$, and $\tilde{\xi} : S(V) \longrightarrow P$ the morphism of operads induced by ξ . Then $P \sqcup_\xi V$ is isomorphic to the push-out of the following diagram of operads

$$\begin{array}{ccc} S(V) & \xrightarrow{\tilde{\xi}} & P \\ i_V \downarrow & & \\ T(V) & & \end{array}$$

If V is concentrated in arity n , and its differential is zero, the operad $P \sqcup_\xi V$ is called an *arity n principal extension*.

Let us explicitly describe it in the case that $n \geq 2$, $P(1) = 0$, and $P \cong t_! t_n P$. First of all, since, for a truncated operad $Q \in \mathbf{Op}(\leq n-1)$, there is a chain of isomorphisms

$$\begin{aligned} \text{Hom}(t_{n-1}(P \sqcup_\xi V), Q) &\cong \text{Hom}(P \sqcup_\xi V, t_* Q) \\ &\cong \text{Hom}(P, t_* Q) \\ &\cong \text{Hom}(t_{n-1} P, Q), \end{aligned}$$

we have $t_{n-1}(P \sqcup_\xi V) \cong t_{n-1} P$. Next, let X be the n -truncated operad extending $t_n P$ defined by

$$X(i) = \begin{cases} P(i), & \text{if } i < n, \\ P(n) \oplus_\xi V, & \text{if } i = n, \end{cases}$$

the composition operations involving V being trivial, because $P(1) = 0$. Then, it is clear that X represents the functor h_ξ restricted to the category $\mathbf{Op}(\leq n)$, so $t_n(P \sqcup_\xi V) \cong X$. Finally, it is easy to check that $t_!X$ satisfies the universal property of $P \sqcup_\xi V$. Summing up, we have proven:

Proposition 4.3.1. *Let $n \geq 2$ an integer. Let P be an operad such that $P(1) = 0$ and $t_!t_nP \cong P$, V a dg Σ -module concentrated in arity n with zero differential, and $\xi : V[-1] \rightarrow P(n)$ a chain map of Σ_n -modules. The principal extension $P \sqcup_\xi V$ satisfies:*

$$(1) \quad t_{n-1}(P \sqcup_\xi V) \cong t_{n-1}P.$$

$$(2) \quad (P \sqcup_\xi V)(n) \cong C(\xi), \text{ in particular, there exists an exact sequence of complexes}$$

$$0 \rightarrow P(n) \rightarrow (P \sqcup_\xi V)(n) \rightarrow V \rightarrow 0.$$

$$(3) \quad P \sqcup_\xi V \cong t_!t_n(P \sqcup_\xi V).$$

$$(4) \quad \text{A morphism of operads } \phi : P \sqcup_\xi V \rightarrow Q \text{ is determined by a morphism of } n\text{-truncated operads } f : t_nP \rightarrow t_nQ, \text{ and a homogeneous map of } \Sigma_n\text{-modules } g : V \rightarrow Q(n), \text{ such that } f\xi = dg.$$

These results extend trivially to truncated operads.

4.4. Minimal objects. Now we can translate the definition of minimality of operads of dg modules in terms of the canonical tower:

Proposition 4.4.1. *An operad M is minimal if, and only if, $M(1) = 0$ and the canonical tower of M*

$$0 = t_!t_1M \rightarrow \cdots \rightarrow t_!t_{n-1}M \rightarrow t_!t_nM \rightarrow \cdots$$

is a sequence of principal extensions.

Proof. Let $M = (\Gamma(V), d_M)$ be a minimal operad. Then (see [MSS], formula II.(3.89))

$$t_!t_nM \cong (\Gamma(V(\leq n)), \partial_n),$$

and $t_!t_nM$ is an arity n principal extension of $t_!t_{n-1}M$ defined by $\partial_n : V(n) \rightarrow (t_!t_{n-1}M)(n)$.

Conversely, let us suppose $M(1) = 0$, and that $t_!t_{n-1}M \rightarrow t_!t_nM$ is an arity n principal extension defined by a Σ -module $V(n)$ concentrated in arity n and zero differential, for each n . Then, $M = \Gamma(\bigoplus_{n \geq 2} V(n))$ and its differential is decomposable, because $M(1) = 0$. So M is a minimal operad. \square

4.4.1. We now give the definition of minimality for truncated operads.

For $m \leq n$ we have an obvious truncation functor

$$t_m : \mathbf{Op}(\leq n) \rightarrow \mathbf{Op}(\leq m),$$

which has a right adjoint t_* and a left adjoint $t_!$.

Definition 4.4.2. A n -truncated operad M is said to be *minimal* if $M(1) = 0$ and the canonical tower

$$0 = t_1 t_1 M \longrightarrow t_1 t_2 M \longrightarrow \cdots \longrightarrow t_1 t_{n-1} M \longrightarrow M$$

is a sequence of (n -truncated) principal extensions.

An operad M is said to be *n -minimal* if the truncation $t_n M$ is minimal.

It follows from the definitions that an operad M is n -minimal if, and only if, $t_1 t_n M$ is minimal. It is clear that an operad M is minimal if and only if it is n -minimal for every n , and that theorems 3.120, 3.123 and 3.125 of [MSS] remain true in $\mathbf{Op}(\leq n)$, merely replacing “minimal” by “ n -minimal”.

4.4.2. The category \mathbf{Op} has a natural structure of closed model category ([Hi]). For our present purposes, we will not need all the model structure, only a small piece: the notion of homotopy between morphisms of operads and the fact that minimal operads are cofibrant objects in \mathbf{Op} ; this can be developed independently, as in [MSS], II.3.10. From these results the next one follows easily.

Proposition 4.4.3. *Let M be a minimal operad and P a suboperad. If the inclusion $P \hookrightarrow M$ is a weak equivalence, then $P = M$.*

Proof. Let us call $i : P \hookrightarrow M$ the inclusion. By [MSS], th. II.3.123, we can lift, up to homotopy, the identity of M in the diagram below

$$\begin{array}{ccc} & P & \\ & \downarrow i & \\ M & \xrightarrow{\text{id}} & M \end{array}$$

So we obtain a morphism of operads $f : M \longrightarrow P$ such that if is homotopic to id . Hence if is a weak equivalence and, by [MSS], prop. II.3.120, it is an isomorphism. Therefore i is an isomorphism too. \square

4.5. **Automorphisms of a formal minimal operad.** For an operad P , let $\text{Aut}(P)$ denote the group of its automorphisms. The following lifting property from automorphisms of the homology of the operad to automorphisms of the operad itself is the first part of the characterization of formality that we will establish in th. 5.2.4.

Proposition 4.5.1. *Let M be a minimal operad. If M is formal, then the map $H : \text{Aut}(M) \longrightarrow \text{Aut}(HM)$ is surjective.*

Proof. Because M is a formal operad, we have a sequence of weak equivalences

$$M \longleftarrow X_1 \longrightarrow X_2 \longleftarrow \cdots \longrightarrow X_{n-1} \longleftarrow X_n \longrightarrow HM.$$

By the lifting property of minimal operads ([MSS], th. II.3.123) there exists a weak equivalence

$$\rho : M \longrightarrow HM.$$

Let $\phi \in \text{Aut}(HM)$. Again by the lifting property of minimal operads, given the diagram

$$\begin{array}{ccc}
& & M \\
& & \downarrow \rho \\
M & \xrightarrow{(H\rho)\phi(H\rho)^{-1}\rho} & HM.
\end{array}$$

there exists a morphism $f : M \longrightarrow M$ such that ρf is homotopic to $(H\rho)\phi(H\rho)^{-1}\rho$. Since homotopic maps induce the same morphism in homology, it turns out that f is a weak equivalence and, by ([MSS] II.th.3.120), it is also an isomorphism, because M is minimal. Finally, from $(H\rho)(Hf) = (H\rho)\phi(H\rho)^{-1}(H\rho)$, $Hf = \phi$ follows. \square

It is clear that prop. 4.5.1 remains true in $\mathbf{Op}(\leq n)$, merely replacing “minimal” and “formal” by “ n -minimal” and “ n -formal”, respectively.

4.6. Finiteness of the minimal model. In this section we will show that we can transfer the finiteness conditions of the homology of an operad to the finiteness conditions of its minimal model.

Definition 4.6.1. A Σ -module V is said to be of *finite type* if, for every l , $V(l)$ is a finite dimensional \mathbf{k} -vector space. An operad P is said to be of *finite type* if the underlying Σ -module, UP , is of finite type.

Example 4.6.2. If V is a Σ -module of finite type such that $V(1) = 0$, then the free operad $\Gamma(V)$ is of finite type, because

$$\Gamma(V)(n) \cong \bigoplus_{T \in \text{Tree}(n)} V(T),$$

where $\text{Tree}(n)$ is the finite set of isomorphism classes of n -labelled reduced trees, and $V(T) = \bigotimes_{v \in \text{Vert}(T)} V(\text{In}(v))$, for every n -labelled tree T , (see [MSS], II.1.84). In particular, if P is a n -truncated operad of finite type, then $t_!P \cong \Gamma(P)/J$ (see prop. 4.2.3) is of finite type as well.

Theorem 4.6.3. *Let P be an operad. If the homology of P is of finite type, then every minimal model P_∞ of P is of finite type.*

Proof. Let M be a minimal operad such that HM is of finite type. Since

$$M(n) = (t_!t_n M)(n),$$

it suffices to check that $t_!t_n M$ is of finite type. We proceed by induction. The first step of the induction is trivial because $M(1) = 0$. Then, $t_!t_n M$ is an arity n principal extension of the operad $t_!t_{n-1}M$ by the vector space

$$V(n) = HC((t_!t_{n-1}M)(n) \rightarrow M(n)),$$

thus $t_n(t_!t_n M)$ is finite dimensional, by the induction hypothesis. Therefore $t_!t_n M = t_!(t_n t_!t_n M)$ is also of finite type, by the previous example. \square

5. WEIGHT THEORY IMPLIES FORMALITY

In analogy to the formality theorem of [DGMS] for the rational homotopy type of a compact Kähler manifold, Deligne ([D]) proved formality of the “ $\overline{\mathbb{Q}_l}$ -homotopy type” of a smooth projective variety defined over a finite field using the weights of the Frobenius action in the l -adic cohomology and his solution of the Riemann hypothesis. In this section we follow this approach and introduce weights to establish a criterion of formality for operads based on the formality of the category of pure complexes, defined below.

In this section \mathbf{k} will denote a field of characteristic zero, and an operad will be an operad in $\mathbf{C}_*(\mathbf{k})$.

5.1. Weights. A *weight function* on \mathbf{k} is a group morphism $w : \Gamma \longrightarrow \mathbb{Z}$ defined on a subgroup Γ of the multiplicative group $\overline{\mathbf{k}}^*$ of an algebraic closure $\overline{\mathbf{k}}$ of \mathbf{k} .

An element $\lambda \in \overline{\mathbf{k}}$ is said to be *pure of weight n* if $\lambda \in \Gamma$ and $w(\lambda) = n$. A polynomial $q(t) \in \mathbf{k}[t]$ is said to be *pure of weight n* if all the roots of $q(t)$ in $\overline{\mathbf{k}}$ are pure of weight n . We will write $w(q) = n$ in this case.

Let f be an endomorphism of a \mathbf{k} -vector space V . If V is of finite dimension, we will say that f is *pure of weight n* if its characteristic polynomial is pure of weight n . When f is understood, we will say that V is pure of weight n .

Let f be an endomorphism of a finite dimensional \mathbf{k} -vector space V . If $q(t) \in \mathbf{k}[t]$ is an irreducible polynomial, we will denote by $\ker q(f)^\infty$ the primary component corresponding to the irreducible polynomial $q(t)$, that is, the union of the subspaces $\ker q(f)^n$, $n \geq 1$. The space V decomposes as a direct sum of primary components $V = \bigoplus \ker q(f)^\infty$, where $q(t)$ runs through the set of all irreducible factors of the minimal polynomial of f . The sum of the primary components corresponding to the pure polynomials of weight n will be denoted by V^n , that is $V^n = \bigoplus_{w(q)=n} \ker q(f)^\infty$. Hence we have a decomposition

$$V = \bigoplus_n V^n \oplus C,$$

where C is the sum of the primary components corresponding to the polynomials which are not pure. This decomposition will be called the *weight decomposition* of V .

This weight decomposition is obviously functorial on the category of pairs (V, f) .

Let P be a complex of \mathbf{k} -vector spaces such that HP is of finite type, that is, $H_i P$ is finite dimensional, for all i . An endomorphism f of P is said to be *pure of weight n* if $H_i(f)$ is pure of weight $n+i$, for all i . In that case, we will say that (P, f) , or simply P , is *pure of weight n* , if the endomorphism f is understood. Obviously, if (P, f) is pure of weight n , so do is (HP, Hf) .

The following example will be useful in the sequel. Take $\alpha \in \mathbf{k}^*$ that is not a root of unity and define $w : \{\alpha^n; n \in \mathbb{Z}\} \longrightarrow \mathbb{Z}$ by $w(\alpha^n) = n$. Let P be a finite type complex with zero differential. Then, the *grading automorphism* ϕ_α of P , defined by $\phi_\alpha = \alpha^i \cdot \text{id}$ on P_i , for all $i \in \mathbb{Z}$, is pure of weight 0.

Let \mathbb{F} be a finite field of characteristic p and q elements, l a prime $\neq p$. In [D], Deligne defined a weight function w on the field \mathbb{Q}_l as follows. If $\iota : \overline{\mathbb{Q}_l} \longrightarrow \mathbb{C}$ is an embedding, and

$$\Gamma = \{\alpha \in \overline{\mathbb{Q}_l} ; \exists n \in \mathbb{Z} \text{ such that } |\iota\alpha| = q^{\frac{n}{2}}\},$$

then $w : \Gamma \longrightarrow \mathbb{Z}$ is defined by $w(\alpha) = n$, if $|\iota\alpha| = q^{\frac{n}{2}}$. The Riemann hypothesis, proved by Deligne, asserts that the Frobenius action is a pure endomorphism (of weight 0) of the étale cohomology $H^*(X, \mathbb{Q}_l)$ of every smooth projective \mathbb{F} -scheme X .

5.2. Formality criterion. Let w be a weight function on \mathbf{k} and denote by $\mathbf{C}_*^w(\mathbf{k})$ the category of couples (P, f) where P is a finite type complex and f is an endomorphism of P which is pure of weight 0.

Theorem 5.2.1. $\mathbf{C}_*^w(\mathbf{k})$ is a formal symmetric monoidal category.

Proof. First of all, $\mathbf{1}$, with the identity, is pure of weight 0. Next, by the Künneth theorem and elementary linear algebra, if (P, f) and (Q, g) are pure complexes of weights n and m respectively, then $(P \otimes Q, f \otimes g)$ is pure of weight $n + m$. Then it is easy to check that $\mathbf{C}_*^w(\mathbf{k})$ has a structure of symmetric monoidal category such that the assignment

$$\mathbf{C}_*^w(\mathbf{k}) \longrightarrow \mathbf{C}_*(\mathbf{k}), \quad (P, f) \mapsto P$$

is a symmetric monoidal functor, and that the functor of homology $H : \mathbf{C}_*^w(\mathbf{k}) \longrightarrow \mathbf{C}_*(\mathbf{k})$ is a symmetric monoidal functor as well.

To prove that $\mathbf{C}_*^w(\mathbf{k})$ is formal we will use the weight function w to define a symmetric monoidal functor T and weak equivalences

$$\mathrm{id}_{\mathbf{C}_*^w(\mathbf{k})} \longleftarrow T \longrightarrow H.$$

Let (P, f) an object $\mathbf{C}_*^w(\mathbf{k})$. Since each P_i is finite dimensional, P_i has a weight decomposition $P_i = C_i \oplus \bigoplus_n P_i^n$. Then, the components of weight n ,

$$P^n := \bigoplus_i P_i^n,$$

form a subcomplex of P , and the same is true for $C := \bigoplus C_i$. So we have a weight decomposition of P as a direct sum of complexes

$$P = C \oplus \bigoplus_n P^n.$$

Taking homology we obtain

$$HP = HC \oplus \bigoplus_n HP^n.$$

Obviously, this decomposition is exactly the weight decomposition of HP . Purity of f implies $HC = 0$ and $H(P^n) = H_n(P)$, for all $n \in \mathbb{Z}$. Hence the inclusion $\bigoplus_n P^n \rightarrow P$ is a weak equivalence.

Next, for every $n \in \mathbb{Z}$, the homology of the complex P^n is concentrated in degree n . So there is a natural way to define a weak equivalence between the complex P^n and its homology $H(P^n)$.

Let $\tau_{\geq n}P^n$ be the canonical truncation in degree n of P^n ,

$$\tau_{\geq n}P^n := Z_n P^n \oplus \bigoplus_{i>n} P_i^n .$$

This is a subcomplex of P^n , and the inclusion $\tau_{\geq n}P^n \rightarrow P^n$ is a weak equivalence. Since $\tau_{\geq n}P^n$ is non trivial only in degrees $\geq n$, and its homology is concentrated in degree n , the canonical projection $\tau_{\geq n}P^n \rightarrow H(P^n)$ is a chain map, which is a weak equivalence.

Define T by

$$TP := \bigoplus_n \tau_{\geq n}P^n .$$

Obviously T is an additive functor, and $TP = \bigoplus_n T(P^n)$. Moreover, T is a subfunctor of the identity functor of $\mathbf{C}^w(\mathbf{k})$, and the canonical projection $TP \rightarrow HP$ is a weak equivalence.

We prove now that T is a symmetric monoidal subfunctor of the identity. Let P and Q be pure complexes of weight 0. Since T is additive, and $\sum_{i+j=n} P^i \otimes Q^j \subset (P \otimes Q)^n$, it suffices to show that $T(P^i) \otimes T(Q^j) \subset T(P^i \otimes Q^j)$. By the Leibnitz rule, we have an inclusion in degree $i+j$:

$$Z_i P^i \otimes Z_j Q^j \subset Z_{i+j}(P^i \otimes Q^j).$$

In the other degrees the inclusion is trivially true. Hence, T being stable by products, it is a symmetric monoidal subfunctor of the identity.

Finally, the projection on the homology $TP \rightarrow HP$ is well defined and obviously compatible with the Künneth morphism, so the canonical projection $T \rightarrow H$ is a monoidal natural transformation. Therefore $C_*^w(\mathbf{k})$ is a formal symmetric monoidal category. \square

Remark 5.2.2. The formality theorem for the current complex, 3.2.1, could be obtained as a corollary from the formality of the full subcategory of $\mathbf{C}_*(\mathbb{R})$ whose objects are the double complexes that satisfy the dd^c -lemma.

Corollary 5.2.3. *Let P be an operad with homology of finite type. If P has a pure endomorphism (with respect to some weight function w), then P is a formal operad.*

Proof. If $P_\infty \rightarrow P$ is a minimal model of P , then P_∞ is an operad of finite type by 4.6.3. From the lifting property ([MSS], 3.123), there exists an induced pure endomorphism f on P_∞ . Thus (P_∞, f) is an operad of $\mathbf{C}_*^w(\mathbf{k})$, and the corollary follows from th. 5.2.1 and prop. 2.5.5. \square

Let P be an operad. Since Σ -actions and compositions \circ_i are homogeneous maps of degree 0, every grading automorphism, with respect to a non root of unit α , is a pure endomorphism of the operad HP .

Theorem 5.2.4. *Let \mathbf{k} a field of characteristic zero, and P an operad with homology of finite type. The following statements are equivalent:*

- (1) P is formal.
- (2) There exists a model P' of P such that $H : \text{Aut}(P') \rightarrow \text{Aut}(HP)$ is surjective.
- (3) There exists a model P' of P and $f \in \text{Aut}(P')$ such that $H(f) = \phi_\alpha$, for some $\alpha \in \mathbf{k}^*$ non root of unity.
- (4) There exists a pure endomorphism f in a model P' of P .

Proof. (1) \Rightarrow (2) is prop. 4.5.1. (2) \Rightarrow (3) and (3) \Rightarrow (4) are obvious. Finally, (4) \Rightarrow (1) is cor. 5.2.3. \square

6. DESCENT OF FORMALITY

In this section \mathbf{k} will denote a field of characteristic zero, and operad will mean an operad in the category $\mathbf{C}_*(\mathbf{k})$, unless another category was mentioned. Using the characterization of formality of th. 5.2.4, we will prove now that formality does not depend on the ground field, if it has zero characteristic.

6.1. Automorphism group of a finite type operad.

6.1.1. Let P be an operad. Restricting the automorphism we have an inverse system of groups $(\text{Aut}(t_n P))_n$ and a morphism of groups $\text{Aut}(P) \longrightarrow \varprojlim \text{Aut}(t_n P)$. Because $P \cong \varprojlim t_* t_n P$, the following lemma is clear.

Lemma 6.1.1. *The morphisms of restriction induce a canonical isomorphism of groups*

$$\text{Aut}(P) \longrightarrow \varprojlim \text{Aut}(t_n P).$$

6.1.2. In order to prove that the group of automorphisms of a finite type operad is an algebraic group, we start by fixing some notations about group schemes. Let $\mathbf{k} \longrightarrow R$ be a commutative \mathbf{k} -algebra. If P is an operad, its extension of scalars $P \otimes_{\mathbf{k}} R$ is an operad in $\mathbf{C}_*(R)$, and the correspondence

$$R \mapsto \mathbf{Aut}(P)(R) = \text{Aut}_R(P \otimes_{\mathbf{k}} R),$$

where Aut_R means the set of automorphisms of operads in $\mathbf{C}_*(R)$, defines a functor

$$\mathbf{Aut}(P) : \mathbf{k}\text{-alg} \longrightarrow \mathbf{Gr},$$

from the category $\mathbf{k}\text{-alg}$ of commutative \mathbf{k} -algebras, to the category \mathbf{Gr} of groups. It is clear that

$$\mathbf{Aut}(P)(\mathbf{k}) = \text{Aut}(P).$$

We will denote by \mathbb{G}_m the *multiplicative group scheme* defined over the ground field \mathbf{k} .

Proposition 6.1.2. *Let P be a truncated operad. If P is of finite type, then*

- (1) $\text{Aut}(P)$ is an algebraic matrix group over \mathbf{k} .
- (2) $\mathbf{Aut}(P)$ is an algebraic affine group scheme over \mathbf{k} , represented by the algebraic matrix group $\text{Aut}(P)$.
- (3) Homology defines a morphism $\mathbf{H} : \mathbf{Aut}(P) \longrightarrow \mathbf{Aut}(HP)$ of algebraic affine group schemes.

Proof. Let P a finite type n -truncated operad. The sum $M = \sum_{l \leq n} \dim P(l)$ is finite, hence $\text{Aut}(P)$ is the closed subgroup of $\mathbf{GL}_M(\mathbf{k})$ defined by the polynomial equations that express the compatibility with the Σ -action, the differential, and the bilinear compositions \circ_i . Thus

$\mathbf{Aut}(P)$ is an algebraic matrix group. Moreover, $\mathbf{Aut}(P)$ is obviously the algebraic affine group scheme represented by the matrix group $\mathbf{Aut}(P)$.

Next, for every commutative \mathbf{k} -algebra R , the map

$$\mathbf{Aut}(P)(R) = \mathbf{Aut}_R(P \otimes_{\mathbf{k}} R) \longrightarrow \mathbf{Aut}_R(HP \otimes_{\mathbf{k}} R) = \mathbf{Aut}(HP)(R)$$

is a morphism of groups and it is natural in R ; thus (3) follows. \square

Theorem 6.1.3. *Let \mathbf{k} be a field of characteristic zero, and P a finite type truncated operad. If P is minimal, then*

$$\mathbf{N} = \ker(\mathbf{H} : \mathbf{Aut}(P) \longrightarrow \mathbf{Aut}(HP))$$

is a unipotent algebraic affine group scheme over \mathbf{k} .

Proof. Since \mathbf{k} has zero characteristic, and $\mathbf{Aut}(P)$, $\mathbf{Aut}(HP)$ are algebraic by prop. 6.1.2, \mathbf{N} is represented by an algebraic matrix group defined over \mathbf{k} (see [Bo]). So it suffices to verify that all elements in $\mathbf{N}(\mathbf{k})$ are unipotent.

Given $f \in \mathbf{N}(\mathbf{k})$, let $P^1 = \ker(f - \text{id})^\infty$ be the primary component of P corresponding to the eigenvalue 1 (see 5.1). Then P^1 is a suboperad of P , and the inclusion $P^1 \hookrightarrow P$ is a weak equivalence. Since P is minimal, it follows from prop. 4.4.3 that $P = P^1$, thus f is unipotent. \square

6.2. A descent theorem. After these preliminaries, let us prove the descent theorem of the formality for operads. In rational homotopy theory, this corresponds to the descent theorem of formality for cdg algebras of Sullivan and Halperin-Stasheff ([Su] and [HS], see also [Mor] and [R])

Theorem 6.2.1. *Let \mathbf{k} be a field of characteristic zero, and $\mathbf{k} \subset \mathbf{K}$ a field extension. If P is an operad in $\mathbf{C}_*(\mathbf{k})$ with homology of finite type, then the following statements are equivalent:*

- (1) P is formal.
- (2) $P \otimes \mathbf{K}$ is a formal operad in $\mathbf{C}_*(\mathbf{K})$.
- (3) For every n , $t_n P$ is formal.

Proof. Because the statements of the theorem only depend on the homotopy type of the operad, we can assume P to be minimal and, by th. 4.6.3, of finite type. Moreover, minimality of P is equivalent to the minimality of all its truncations, $t_n P$.

Let us consider the following additional statement:

- (2 $_{\frac{1}{2}}$) For every n , $t_n P \otimes \mathbf{K}$ is formal.

We will prove the following sequence of implications

$$(1) \Rightarrow (2) \Rightarrow (2_{\frac{1}{2}}) \Rightarrow (3) \Rightarrow (1),$$

(1) implies (2) because $_{-} \otimes_{\mathbf{k}} \mathbf{K}$ is an exact functor.

If $P \otimes \mathbf{K}$ is formal, then so are all of its truncations $t_n(P \otimes \mathbf{K}) \cong t_n P \otimes \mathbf{K}$, because truncation functors are exact, so (2) implies (2 $_{\frac{1}{2}}$).

Let us see that $(2\frac{1}{2})$ implies (3). From the implication $(1) \Rightarrow (2)$, already proven, it is clear that we may assume \mathbf{K} to be algebraically closed. So, let \mathbf{K} be an algebraically closed field, n an integer, and P a finite type minimal operad such that $t_n P \otimes \mathbf{K}$ is formal. Since

$$\mathbf{Aut}(t_n P)(\mathbf{K}) \longrightarrow \mathbf{Aut}(Ht_n P)(\mathbf{K})$$

is a surjective map, by th. 5.2.4, it results that

$$\mathbf{Aut}(t_n P) \longrightarrow \mathbf{Aut}(Ht_n P)$$

is a quotient map. Thus, by ([Wa], 18.1), we have an exact sequence of groups

$$1 \longrightarrow \mathbf{N}(\mathbf{k}) \longrightarrow \mathbf{Aut}(t_n P)(\mathbf{k}) \longrightarrow \mathbf{Aut}(Ht_n P)(\mathbf{k}) \longrightarrow H^1(\mathbf{K}/\mathbf{k}, \mathbf{N}) \longrightarrow \dots$$

Since \mathbf{N} is unipotent by th. 6.1.3, and \mathbf{k} has zero characteristic, it follows that $H^1(\mathbf{K}/\mathbf{k}, \mathbf{N})$ is trivial (*op. cit.*, 18.2.e). So we have an exact sequence of groups

$$1 \rightarrow \mathbf{N}(\mathbf{k}) \longrightarrow \mathbf{Aut}(t_n P) \longrightarrow \mathbf{Aut}(Ht_n P) \longrightarrow 1$$

In particular, $\mathbf{Aut}(t_n P) \longrightarrow \mathbf{Aut}(Ht_n P)$ is surjective. Hence, again by th. 5.2.4, $t_n P$ is a formal operad.

Let us see finally that (3) implies (1). By th. 5.2.4, it suffices to prove that all the grading automorphisms have a lift. Let $\phi : \mathbb{G}_m \longrightarrow \mathbf{Aut}(Ht_n P)$ the grading representation that sends $\alpha \in \mathbb{G}_m$ to the grading automorphism ϕ_α defined in 5.1. For every n , form the pull-back of algebraic affine group schemes:

$$\begin{array}{ccc} \mathbf{F}_n & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \phi \\ \mathbf{Aut}(t_n P) & \xrightarrow{H} & \mathbf{Aut}(Ht_n P) \end{array}$$

That is to say, for every commutative \mathbf{k} -algebra R ,

$$\mathbf{F}_n(R) = \{(f, \alpha) \in \mathbf{Aut}(t_n P)(R) \times \mathbb{G}_m(R) ; Hf = \phi_\alpha\} .$$

By 6.1.1, we have a commutative diagram

$$\begin{array}{ccc} \varprojlim \mathbf{F}_n(\mathbf{k}) & \longrightarrow & \mathbb{G}_m(\mathbf{k}) \\ \downarrow & & \downarrow \phi \\ \mathbf{Aut}(P) \cong \varprojlim \mathbf{Aut}(t_n P) & \xrightarrow{H} & \mathbf{Aut}(HP) \cong \varprojlim \mathbf{Aut}(Ht_n P) \end{array}$$

so, to lift grading automorphisms, it suffices to verify that the map $\varprojlim \mathbf{F}_n(\mathbf{k}) \longrightarrow \mathbb{G}_m(\mathbf{k})$ is surjective. In order to prove this surjectivity, first we will replace the inverse system $(\mathbf{F}_n(\mathbf{k}))_n$ by an inverse system $(\mathbf{F}'_n(\mathbf{k}))_n$ whose transition maps are surjective. Indeed, for all $p \geq n$, the restriction $\varrho_{p,n} : \mathbf{F}_p \longrightarrow \mathbf{F}_n$ is a morphism of algebraic affine group schemes which are represented by algebraic matrix groups, so, by ([Wa], 15.1), it factors as a quotient map and a closed embedding:

$$\mathbf{F}_p \longrightarrow \mathrm{im} \varrho_{p,n} \longrightarrow \mathbf{F}_n .$$

Denote $\mathbf{F}'_n := \bigcap_{p \geq n} \mathrm{im} \varrho_{p,n}$. Since $\{\mathrm{im} \varrho_{p,n}\}_{p \geq n}$ is a descending chain of closed subschemes of the noetherian scheme \mathbf{F}_n , there exists an integer $N(n) \geq n$ such that

$$\mathbf{F}'_n = \mathrm{im} \varrho_{N(n),n} ,$$

thus the restrictions $\varrho_{n+1,n}$ induce quotient maps $\varrho_{n+1,n} : \mathbf{F}'_{n+1} \longrightarrow \mathbf{F}'_n$. So, applying again ([Wa], 18.1), we have an exact sequence of groups

$$1 \longrightarrow \mathbf{N}'(\mathbf{k}) \longrightarrow \mathbf{F}'_{n+1}(\mathbf{k}) \longrightarrow \mathbf{F}'_n(\mathbf{k}) \longrightarrow H^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{N}) \longrightarrow \dots$$

Here, $\mathbf{N}'(\mathbf{k})$ is a closed subscheme of $\mathbf{N}(\mathbf{k})$ because, for every $(f, \alpha) \in \mathbf{N}'(\mathbf{k})$ we have $\alpha = 1$ and so $Hf = 1$ in $Ht_{\leq n+1}^*P$, which means that $f \in \mathbf{N}(\mathbf{k})$. By th. 6.1.3, $\mathbf{N}'(\mathbf{k})$ is unipotent, thus, as in the previous implication, it follows that $\mathbf{F}'_{n+1}(\mathbf{k}) \longrightarrow \mathbf{F}'_n(\mathbf{k})$ is surjective for all $n \geq 2$.

Since in the inverse system $(\mathbf{F}'_n(\mathbf{k}))_n$ all the transition maps are surjective, the map

$$\lim_{\leftarrow} \mathbf{F}'_p(\mathbf{k}) \longrightarrow \mathbf{F}'_2(\mathbf{k}) .$$

is surjective as well. Moreover, $\mathbf{F}'_2(\mathbf{k}) \longrightarrow \mathbb{G}_m(\mathbf{k})$ is also surjective. Indeed, given $\alpha \in \mathbb{G}_m(\mathbf{k})$, since $t_{N(2)}P$ is formal by hypothesis, by th. 5.2.4 we can lift the grading automorphism $\phi_\alpha \in \text{Aut}(Ht_{N(2)}P)$ to an automorphism $f \in \text{Aut}(t_{N(2)}P)$. So we have an element $(f, \alpha) \in \mathbf{F}_{N(2)}(\mathbf{k})$, whose image in $\mathbf{F}_2(\mathbf{k})$ will be an element of $\mathbf{F}'_2(\mathbf{k})$ which will project onto α .

We conclude that $\lim_{\leftarrow} \mathbf{F}'_p(\mathbf{k}) \longrightarrow \mathbb{G}_m(\mathbf{k})$ is surjective, hence P is formal. \square

6.3. Applications. As an immediate consequence of th. 6.2.1, the previous theorems 3.3.1, and 3.4.2 of formality over \mathbb{R} imply, respectively, the following corollaries

Corollary 6.3.1. *If X is an operad in $\mathbf{K\ddot{a}h}$, then the operad of cubic chains $C_*(X; \mathbb{Q})$ is formal.*

Corollary 6.3.2. *If X is an operad in $\mathbf{DMV}(\mathbb{C})$, then the operad of cubic chains $C_*(X; \mathbb{Q})$ is formal.*

Finally, we can apply th. 6.2.1 to the formality of the little k -disc operad. Let \mathcal{D}_k denote the *little k -discs* operad of Boardman and Vogt. It is the topological operad with $\mathcal{D}_k(1) = \mathbf{pt}$, and, for $l \geq 2$, $\mathcal{D}_k(l)$ is the space of configurations of l disjoint discs inside the unity disc of \mathbb{R}^k .

M. Kontsevich proved that the operad of cubic chains $C_*(\mathcal{D}_k, \mathbb{R})$ is formal ([Ko]). Therefore, from th. 6.2.1, we obtain

Corollary 6.3.3. *The operad of cubic chains of the little k -discs operad $C_*(\mathcal{D}_k; \mathbb{Q})$ is formal.*

7. CYCLIC OPERADS

7.1. Basic results. Let us recall some definitions from [GeK94] (see also [GeK98] and [MSS]). For all $l \in \mathbb{N}$, the group $\Sigma_l^+ := \text{Aut}\{0, 1, \dots, l\}$ contains Σ_l as a subgroup, and it is generated by Σ_l and the cyclic permutation of order $l + 1$, $\tau_l : (0, 1, \dots, l) \mapsto (1, 2, \dots, l, 0)$.

Let \mathcal{C} be a symmetric monoidal category. A *cyclic* Σ -module E in \mathcal{C} is a sequence $(E(l))_{l \geq 1}$ of objects of \mathcal{C} together with an action of Σ_l^+ on each $E(l)$. Let $\Sigma^+\mathbf{Mod}$ denote the category of cyclic Σ -modules. Forgetting the action of the cyclic permutation we have a functor

$$U^- : \Sigma^+\mathbf{Mod} \longrightarrow \Sigma\mathbf{Mod} .$$

A *cyclic operad* is a cyclic Σ -module P whose underlying Σ -module U^-P has the structure of an operad compatible with the action of the cyclic permutation (see *loc. cit.*). Let \mathbf{Op}^+ denote the category of cyclic operads. We also have an obvious forgetful functor

$$U^- : \mathbf{Op}^+ \longrightarrow \mathbf{Op}.$$

There are obvious extensions of the notions of free operad, homology, weak equivalence, minimality and formality for cyclic dg operads and all the results in the previous sections can be easily transferred to the cyclic setting. In particular, every cyclic dg operad P with $HP(1) = 0$, has a minimal model P_∞ . Moreover $U^-(P_\infty)$ is a minimal model of $U^-(P)$. Finally, we can deduce results analogous to the formality criterion (th. 5.2.4), and to the descent of formality (th. 6.2.1) for cyclic operads.

Let \mathcal{A} an abelian category. It is clear that a formal symmetric monoidal functor $F : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathcal{A})$ induces a functor of cyclic operads

$$F : \mathbf{Op}_{\mathcal{C}}^+ \longrightarrow \mathbf{Op}_{\mathbf{C}_*(\mathcal{A})}^+$$

which sends cyclic operads in \mathcal{C} to formal cyclic operads in $\mathbf{C}_*(\mathcal{A})$. From theorems 3.2.3 and 6.2.1 it follows that $C_*(X; \mathbb{Q})$ is a formal cyclic operad, for every cyclic operad X in **Käh**.

7.2. Formality of the cyclic operad $C_*(\overline{\mathcal{M}}_0; \mathbb{Q})$. Let us apply the previous results to the configuration operad.

Let $\mathcal{M}_{0,l}$ be the moduli space of l different labelled points on the complex projective line \mathbb{P}^1 . For $l \geq 3$, let $\overline{\mathcal{M}}_{0,l}$ denote its *Grothendieck-Knudsen compactification*, that is, the moduli space of stable curves of genus 0, with l different labelled points.

For $l = 1$, put $\overline{\mathcal{M}}_0(1) = *$, a point, and for $l \geq 2$, let $\overline{\mathcal{M}}_0(l) = \overline{\mathcal{M}}_{0,l+1}$. The family of spaces $\overline{\mathcal{M}}_0 = (\overline{\mathcal{M}}_0(l))_{l \geq 1}$ is a cyclic operad in $\mathbf{V}(\mathbb{C})$ ([GiK], or [MSS]). Applying the functor of cubic chains componentwise we obtain a dg cyclic operad $C_*(\overline{\mathcal{M}}_0; \mathbb{Q})$. So, we have the following results.

Corollary 7.2.1. *The cyclic operad of cubic chains $C_*(\overline{\mathcal{M}}_0; \mathbb{Q})$ is formal.*

Corollary 7.2.2. *The categories of strongly homotopy $C_*(\overline{\mathcal{M}}_0; \mathbb{Q})$ and $H_*(\overline{\mathcal{M}}_0; \mathbb{Q})$ -algebras are equivalent.*

8. MODULAR OPERADS

8.1. Preliminaries. Let us recall some definitions and notations about modular operads (see [GeK98], or [MSS], for details).

8.1.1. Let \mathcal{C} be a symmetric monoidal category. A *modular Σ -module* of \mathcal{C} is a bigraded object of \mathcal{C} , $E = (E((g, l)))_{g, l}$, with $g, l \geq 0$, $2g - 2 + l > 0$, such that $E((g, l))$ has a right Σ_l -action. Let us denote the category of modular Σ -modules by $\mathbf{MMod}_{\mathcal{C}}$, or just \mathbf{MMod} if no confusion can arise.

8.1.2. A *modular operad* is a modular Σ -module P , together with *composition* morphisms

$$\circ_i : P((g, l)) \otimes P((h, m)) \longrightarrow P((g + h, l + m - 2)) , \ 1 \leq i \leq l,$$

and *contraction* morphisms

$$\xi_{ij} : P((g, l)) \longrightarrow P((g + 1, l - 2)) , \ 1 \leq i \neq j \leq l,$$

which verify axioms of associativity, commutativity and compatibility (see [GeK98], [MSS]). Let us denote the category of modular operads by $\mathbf{MOp}_{\mathcal{C}}$, or just \mathbf{MOp} if no confusion can arise.

As for operads and cyclic operads, from the definitions it follows that every symmetric monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ applied componentwise induces a functor

$$\mathbf{MOp}_F : \mathbf{MOp}_{\mathcal{C}} \longrightarrow \mathbf{MOp}_{\mathcal{D}} ,$$

and every monoidal natural transformation $\phi : F \Rightarrow G$ between symmetric monoidal functors induces a natural transformation $\mathbf{MOp}_{\phi} : \mathbf{MOp}_F \Rightarrow \mathbf{MOp}_G$.

Example 8.1.1. As Getzler and Kapranov proved ([GeK98]), the family $\overline{\mathcal{M}}((g, l)) := \overline{\mathcal{M}}_{g, l}$ of Deligne-Knudsen-Mumford moduli spaces of stable genus g algebraic curves with l marked points, with the maps that identify marked points, is a modular operad in the category of projective smooth DM-stacks.

8.2. **dg modular operads.** From now on, \mathbf{k} will denote a field of characteristic zero, and modular operads in $\mathbf{C}_*(\mathbf{k})$ will be called simply dg modular operads.

Example 8.2.1. Let V be a finite type chain complex of \mathbf{k} -vector spaces, and B an inner product over V , that is to say, a non-degenerate graded symmetric bilinear form $B : V \otimes V \longrightarrow \mathbf{k}$ of degree 0. It is shown in [GeK98] that there exists a dg modular operad $\mathcal{E}[V]$ such that

$$\mathcal{E}[V]((g, l)) = V^{\otimes l},$$

with the obvious structure morphisms.

An *ideal* of a dg modular operad P is a modular Σ -submodule I of P , such that $P \cdot I \subset I$, $I \cdot P \subset I$, and I is closed under the contractions ξ_{ij} .

For any dg modular operad P and any ideal I of P , the quotient P/I , inherits a natural structure of dg modular operad and the projection $P \longrightarrow P/I$ is a morphism of dg modular operads.

If P is a dg modular operad its *homology* HP , defined by $(HP)((g, l)) = H(P((g, l)))$, is also a dg modular operad. A morphism $\rho : P \longrightarrow Q$ of dg modular operads is said to be a *weak equivalence* if $\rho((g, l)) : P((g, l)) \longrightarrow Q((g, l))$ is a weak equivalence for all (g, l) .

The localization of $\mathbf{MOp}_{\mathbf{C}_*(\mathbf{k})}$ with respect to the weak equivalences is denoted by $\mathbf{HoMOp}_{\mathbf{C}_*(\mathbf{k})}$.

Definition 8.2.2. A dg modular operad P is said to be *formal* if P is weakly equivalent to its homology HP .

Clearly, for a formal symmetric monoidal functor $F : \mathcal{C} \longrightarrow \mathbf{C}_*(\mathbf{k})$, the induced functor

$$F : \mathbf{MOp}_{\mathcal{C}} \longrightarrow \mathbf{MOp}_{\mathbf{C}_*(\mathbf{k})}$$

transforms modular operads in \mathcal{C} to formal modular operads in $\mathbf{C}_*(\mathbf{k})$.

8.3. Modular dimension. In order to study the homotopy properties of dg modular operads we will replace the arity truncation with the truncation with respect to the modular dimension.

Let \mathcal{C} be a symmetric monoidal category.

Recall that, the dimension as algebraic variety of the moduli space $\overline{\mathcal{M}}_{g,l}$ is $3g - 3 + l$. So the following definition is a natural one. The function $d : \mathbb{Z}^2 \rightarrow \mathbb{Z}$, given by $d(g, l) = 3g - 3 + l$, will be called the *modular dimension* function.

Definition 8.3.1. Let E be a modular Σ -module in \mathcal{C} . The modular dimension function induces a graduation $(E_n)_{n \geq 0}$ on E by $E_n = (E((g, l)))_{d(g, l) = n}$, and a decreasing filtration $(E_{\geq n})_n$ of E by

$$E_{\geq n} := (E((g, l)))_{d(g, l) \geq n} .$$

The following properties are easily checked.

Proposition 8.3.2. *Let P be a modular operad in \mathcal{C} . The modular dimension grading satisfies*

$$P \cdot P_n \subset P_{\geq n+1},$$

where $P \cdot P_n$ is the set of meaningful products $\alpha \circ_i \beta$, with $\alpha \in P_m$, $\beta \in P_l$ and at least one of l, m is n . On the other hand, the contraction maps satisfies

$$\xi_{ij} : P_n \longrightarrow P_{\geq n+1} ,$$

for all i, j .

8.4. Truncation of modular operads.

Definition 8.4.1. A *n-truncated modular operad* in a symmetric monoidal category \mathcal{C} is a modular operad defined only up to modular dimension n , that is, a family of \mathcal{C} , $\{P((g, l)); g, l \geq 0, 2g - 2 + l > 0, d(g, l) \leq n\}$, such that $P((g, l))$ has a right Σ_l -action, with morphisms

$$\circ_i : P((g, l)) \otimes P((h, m)) \longrightarrow P((g + h, l + m - 2)) , \quad 1 \leq i \leq l,$$

and *contractions*

$$\xi_{ij} : P((g, l)) \longrightarrow P((g + 1, l - 2)) , \quad 1 \leq i \neq j \leq l,$$

satisfying those axioms in **MOp** that make sense.

If **MOp** $_{\leq n}$ denotes the category of *n-truncated dg modular operads*, we have a *modular dimension truncation* functor,

$$t_n : \mathbf{MOp} \longrightarrow \mathbf{MOp}_{\leq n} ,$$

defined by $t_n(P) = (P((g, l)))_{d(g, l) \leq n}$.

Since the obvious forgetful functor

$$U : \mathbf{MOp} \longrightarrow \mathbf{MMod}$$

has a left adjoint, the *free modular operad functor*

$$\mathbb{M} : \mathbf{MMod} \longrightarrow \mathbf{MOp} ,$$

(see[GeK98], 2.18), by prop. 8.3.2 we can translate the truncation formalism developed in 4.2 to the setting of dg modular operads. So we have a sequence of adjunctions $t_! \dashv t_n \dashv t_*$, and the propositions 4.2.2 and 4.2.3 are still true, merely replacing “operad” with “modular operad”, and “arity” with “modular dimension” shifted by +2. For instance, the arity truncation begins with t_2 , whereas the modular dimension truncation begins with t_0 .

If P is a dg modular operad, the direct system of dg modular operads given by

$$0 \longrightarrow t_! t_0 P \longrightarrow \cdots \longrightarrow t_! t_{n-1} P \longrightarrow t_! t_n P \longrightarrow \cdots$$

is called the *canonical tower* of P .

8.5. Principal extensions. Let us explicitly describe the construction of a principal extension in the context of modular operads. Let P be a dg modular operad, V a dg modular Σ -module with zero differential, concentrated in modular dimension $n \geq 0$, and $\xi : V[-1] \longrightarrow P_n$ a chain map. Then the principal extension of P by ξ , $P \sqcup_\xi V$, is defined by a universal property as in 4.3.1,

$$\mathrm{Hom}_{\mathbf{MOP}}(P \sqcup_\xi V, Q) = \{ (f, g); f \in \mathrm{Hom}_{\mathbf{MOP}}(P, Q), g \in \mathrm{Hom}_{\mathbf{GrMMod}}(V, UQ)_0, d_Q g - g d_V = f \xi \}.$$

In particular we have

$$(P \sqcup_\xi V)_i = \begin{cases} P_i, & \text{if } i < n, \\ P_n \oplus_\xi V, & \text{if } i = n, \end{cases}$$

because in $t_n(P \sqcup_\xi V)$ all the structural morphisms involving V are trivial, by prop. 8.3.2.

Furthermore, the following property, analogous to prop. 4.3.1, is satisfied.

Proposition 8.5.1. *Let $n \geq 0$ an integer. Let P be a dg modular operad such that $t_! t_n P \cong P$, V a dg modular Σ -module concentrated in modular dimension n , with zero differential, and $\xi : V[-1] \longrightarrow P_n$ a morphism of dg modular Σ_n -modules. The principal extension $P \sqcup_\xi V$ satisfy*

$$(1) \quad t_{n-1}(P \sqcup_\xi V) \cong t_{n-1}P.$$

$$(2) \quad (P \sqcup_\xi V)_n \cong C\xi, \text{ in particular, there exists an exact sequence of complexes}$$

$$0 \longrightarrow P_n \longrightarrow (P \sqcup_\xi V)_n \longrightarrow V \longrightarrow 0.$$

$$(3) \quad P \sqcup_\xi V \cong t_! t_n(P \sqcup_\xi V).$$

(4) *A morphism of dg modular operads $\phi : P \sqcup_\xi V \longrightarrow Q$ is determined by a morphism of n -truncated dg modular operads $f : t_n P \longrightarrow t_n Q$, and a homogeneous map $g : V \longrightarrow Q_n$ of modular Σ -modules, such that $f\xi = dg$.*

8.6. Minimal models.

8.6.1. Minimal objects.

Definition 8.6.1. A dg modular operad M is said to be *minimal* if the canonical tower

$$0 \longrightarrow t_! t_0 M \longrightarrow \cdots \longrightarrow t_! t_{n-1} M \longrightarrow t_! t_n M \longrightarrow \cdots$$

is a sequence of principal extensions.

A *minimal model* of a dg modular operad P is a minimal dg modular operad P_∞ together with a weak equivalence $P_\infty \longrightarrow P$ in \mathbf{MOp} .

From prop. 8.5.1, it follows, by induction on the modular dimension, that:

Proposition 8.6.2. *Let M, N be minimal dg modular operads. If $\rho : M \longrightarrow N$ is a weak equivalence of dg modular operads, then ρ is an isomorphism.*

8.6.2. Existence of minimal models.

Theorem 8.6.3. *Let \mathbf{k} be a field of characteristic zero. Every modular operad P in $\mathbf{C}_*(\mathbf{k})$ has a minimal model.*

Proof. We start in modular dimension 0. Let $M^0 = \mathbb{M}HP_0$, and $s : HP_0 \longrightarrow ZP_0$ a section of the canonical projection. Then s induces a morphism of modular operads

$$\rho^0 : M^0 \longrightarrow P$$

which is a weak equivalence of modular operads up to modular dimension 0, because $M_0^0 = HP_0$.

For $n \geq 1$, assume that we have already constructed a morphism of modular operads

$$\rho^{n-1} : M^{n-1} \longrightarrow P$$

such that

- (1) $M^{n-1} \cong t_{|t_{n-1}} M^{n-1}$ is a minimal modular operad, and
- (2) $t_{n-1}(\rho^{n-1})$ is a weak equivalence.

To define the next step of the induction we will use the following statement, which contains the main homological part of the inductive construction of minimal models.

Lemma 8.6.4. *Let*

$$\begin{array}{ccccc} B & & \xrightarrow{\eta} & A & \\ \lambda \downarrow & & & \downarrow \mu & \\ (C\zeta)[-1] & \xrightarrow{-p_Y} & Y & \xrightarrow{\zeta} & X \end{array}$$

be a commutative diagram of complexes of an additive category, then there exists a chain map $\nu : C\eta \longrightarrow X$ such that in the diagram

$$(8.6.3.1) \quad \begin{array}{ccccccc} B & \xrightarrow{\eta} & A & \longrightarrow & C\eta & \longrightarrow & B[1] \\ \lambda \downarrow & & \downarrow \mu & & \downarrow \nu & & \downarrow \lambda[1] \\ (C\zeta)[-1] & \xrightarrow{-p_Y} & Y & \xrightarrow{\zeta} & X & \longrightarrow & C\zeta \end{array}$$

the central square is commutative, and the right hand side square is homotopy commutative. Moreover, the rows of (8.6.3.1) are distinguished triangles, and the vertical maps define a morphism of triangles in the derived category.

We have $C\eta = A \oplus_\eta B[1]$, and $C\zeta = X \oplus_\zeta Y[1]$. Let (λ_X, λ_Y) be the components of λ , then one can check that $\nu(a, b) = \lambda_X(b) + \zeta\mu(a)$, with the homotopy $h(a, b) = (0, \mu(a))$, satisfies the conditions of the statement.

The upper row of the diagram (8.6.3.1) is obviously a distinguished triangle. By axiom $(TR2)$ of a triangulated category, turning the distinguished triangle

$$Y \xrightarrow{\zeta} X \longrightarrow C\zeta \xrightarrow{p_Y} Y[1]$$

one step to the left we obtain that the lower row of the diagram (8.6.3.1) is also a distinguished triangle.

Now we return to the proof of the theorem. Since \mathbf{k} is a field of zero characteristic, the category of modular Σ -modules is semisimple, and $C\rho_n^{n-1}$ is a formal complex of modular Σ -modules. Therefore, if $V = HC\rho_n^{n-1}$, with the zero differential, there exists a weak equivalence

$$s : V \longrightarrow C\rho_n^{n-1}.$$

In fact, s can be obtained from a Σ -equivariant section of the canonical projection from cycles to homology.

Let ξ be the composition

$$V[-1] \xrightarrow{s[-1]} (C\rho_n^{n-1})[-1] \xrightarrow{-p} M_n^{n-1},$$

where the second arrow is the opposite of the canonical projection. We have a commutative diagram of complexes

$$\begin{array}{ccc} V[-1] & \xrightarrow{\xi} & M_n^{n-1} \\ s[-1] \downarrow & & \downarrow \text{id} \\ (C\rho_n^{n-1})[-1] & \xrightarrow{-p} & M_n^{n-1} \xrightarrow{\rho_n^{n-1}} P_n. \end{array}$$

By the previous lemma, there exists a chain map

$$\nu : C\xi \longrightarrow P_n$$

that completes the previous diagram in a diagram

$$(8.6.3.2) \quad \begin{array}{ccccccc} V[-1] & \xrightarrow{\xi} & M_n^{n-1} & \longrightarrow & C\xi & \longrightarrow & V \\ s[-1] \downarrow & & \text{id} \downarrow & & \nu \downarrow & & \downarrow s \\ (C\rho_n^{n-1})[-1] & \xrightarrow{-p} & M_n^{n-1} & \xrightarrow{\rho_n^{n-1}} & P_n & \longrightarrow & C\rho_n^{n-1}. \end{array}$$

where the rows are distinguished triangles in the category of complexes, the central square is commutative, and the vertical maps define a morphism of triangles in the derived category.

The step M^n is defined as the principal extension of M^{n-1} by the attachment map $\xi : V[-1] \longrightarrow M_n^{n-1}$,

$$M^n := M^{n-1} \sqcup_\xi V.$$

Let $\nu_V : V \longrightarrow P$ be the graded map

$$V \longrightarrow C\xi \xrightarrow{\nu} P_n$$

where the first map is the canonical inclusion. Since $\rho^{n-1}\xi = d\nu_V$, the maps ρ^{n-1} and ν_V define, according to the universal property of $M^n = M^{n-1} \sqcup_\xi V$, a morphism of modular operads

$$\rho^n : M^n \longrightarrow P$$

such that $t_{n-1}\rho^n = t_{n-1}\rho^{n-1}$ and $\rho^n = \nu$. By the inductive hypothesis, ρ^n is a weak equivalence in modular dimensions $< n$. Finally, in the diagram (8.6.3.2) s is a weak equivalence, hence ν is a weak equivalence as well. It follows that $t_n\rho^n$ is a weak equivalence, which finishes the induction. Therefore, $\lim_{\rightarrow} M^n$ is a minimal model of P . \square

8.6.3. Finiteness of minimal models.

Definition 8.6.5. A modular Σ -module V is said to be of *finite type* if, for every (g, l) , $V((g, l))$ is a finite dimensional \mathbf{k} -vector space. A dg modular operad P is said to be of *finite type* if UP is of finite type.

Obviously, for every integer $n \geq 0$, there are only a finite number of pairs (g, l) such that $g, l, 2g - 2 + l > 0$ and $d(g, l) = n$, thus a modular Σ -module V is of finite type if, and only if, V_n is finite dimensional, for every $n \geq 0$.

Proposition 8.6.6. *If V is a modular Σ -module of finite type, then $\mathbb{M}(V)$ is of finite type.*

Proof. Indeed, for every pair (g, l) , there is an isomorphism

$$\mathbb{M}(V)((g, l)) \cong \bigoplus_{\gamma \in \{\Gamma((g, l))\}} V((\gamma))_{\text{Aut}(\gamma)}$$

where $\{\Gamma((g, l))\}$ denotes the set of equivalence classes of isomorphisms of stable l -labelled graphs of genus g , the subscript $\text{Aut}(\gamma)$ denotes the space of coinvariants, and

$$V((\gamma)) = \bigotimes_{v \in \text{Vert}(\gamma)} V((g(v), \text{Leg}(v))).$$

By [GeK98] lemma 2.16, the set $\{\Gamma((g, l))\}$ is finite, for every pair (g, l) . Therefore the free modular operad $\mathbb{M}(V)$ is of finite type. \square

As a consequence of prop. 8.6.6 we obtain the finiteness result analogous to 4.6.3.

Theorem 8.6.7. *Let P be a dg modular operad. If HP is of finite type, then every minimal model of P is of finite type.*

8.7. Lifting properties. Analogous to def. II.3.121 of [MSS], there exists a similarly defined path object and a notion of homotopy in the category of dg modular operads.

8.7.1. Homotopy. Let $\mathbf{I} := \mathbf{k}[t, \delta t]$ be the differential graded commutative \mathbf{k} -algebra generated by a generator t in degree 0 and its differential δt in degree -1 . For every dg modular operad P , the path object of P is the dg modular operad $P \otimes \mathbf{I}$, obtained by extension of scalars.

The evaluations at 0 and 1 define two morphisms of modular operads $\rho_0, \rho_1 : P \otimes \mathbf{I} \rightrightarrows P$ which are weak equivalences. An *elementary homotopy* between two morphisms of dg modular operads

$f_0, f_1 : P \rightrightarrows Q$ is a morphism $H : P \longrightarrow Q \otimes \mathbf{I}$ of dg modular operads such that $\rho_i H = f_i$, for $i = 0, 1$. Elementary homotopy is a reflexive and symmetric relation, and the *homotopy relation* between morphisms is the equivalence relation generated by elementary homotopy. Homotopic morphisms induce the same morphism in \mathbf{HoMOp} .

8.7.2. *Lifting properties of minimal objects.* Obstruction theory, that is, lemma II.3.139 *op. cit.*, and its consequences: the homotopy properties of the minimal objects (theorems II.3.120 and II.3.123 *op. cit.*), is easily established in the context of modular operads. So we have

Lemma 8.7.1. *Let $\rho : Q \longrightarrow R$ be a weak equivalence of dg modular operads, and $\iota : P \longrightarrow P \sqcup_\xi V$ a principal extension. For every homotopy commutative diagram in \mathbf{MOp}*

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ \iota \downarrow & & \rho \downarrow \\ P \sqcup_\xi V & \xrightarrow{\psi} & R \end{array}$$

there exists an extension $\bar{\phi} : P \sqcup_\xi V \longrightarrow Q$ of ϕ such that $\rho \bar{\phi}$ is homotopic to ψ . Moreover, $\bar{\phi}$ is unique up to homotopy.

From this lemma the lifting property of minimal modular operads follows by induction:

Theorem 8.7.2. *Let $\rho : Q \longrightarrow R$ be a weak equivalence of dg modular operads, and M a minimal modular operad. For every morphism $\psi : M \longrightarrow R$, there exists a morphism $\tilde{\psi} : M \longrightarrow Q$ such that $\rho \tilde{\psi}$ is homotopic to ψ . Moreover, $\tilde{\psi}$ is unique up to homotopy.*

8.7.3. *Uniqueness of minimal models.* From th. 8.7.2 and prop. 8.6.2, we obtain

Theorem 8.7.3. *Two minimal models of a modular operad are isomorphic.*

The modular analogue of prop. 4.4.3 follows in the same way.

Proposition 8.7.4. *Let M be a minimal dg modular operad and P a subobject of M . If the inclusion $P \hookrightarrow M$ is a weak equivalence, then $P = M$.*

8.8. **Formality.** From theorems 8.7.2 and 8.6.2, the modular analogue of prop. 4.5.1 follows easily.

Proposition 8.8.1. *Let M be a minimal dg modular operad. If M is formal, then the map $H : \text{Aut}(M) \longrightarrow \text{Aut}(HM)$ is surjective.*

Now, from th. 5.2.3 and prop. 8.8.1, the formality criterion for modular operads follows with the same proof as th. 5.2.4.

Theorem 8.8.2. *Let \mathbf{k} be a field of characteristic zero, and P a dg modular operad with homology of finite type. The following statements are equivalent:*

- (1) P is formal.

- (2) *There exists a model P' of P such that $H : \text{Aut}(P') \longrightarrow \text{Aut}(HP)$ is surjective.*
- (3) *There exists a model P' of P , and $f \in \text{Aut}(P')$ such that $Hf = \phi_\alpha$, for some $\alpha \in \mathbf{k}^*$ non root of unity.*
- (4) *There exists a pure endomorphism f in a model P' of P .*

Then, using this result, the descent of formality for modular operads follows as th. 6.2.1.

Theorem 8.8.3. *Let \mathbf{k} be a field of characteristic zero, and $\mathbf{k} \subset \mathbf{K}$ a field extension. If P is a modular operad in $\mathbf{C}_*(\mathbf{k})$ with homology of finite type, then P is formal if, and only if, $P \otimes \mathbf{K}$ is a formal modular operad in $\mathbf{C}_*(\mathbf{K})$.*

Finally, the result below follows from 8.2, and theorems 3.4.1, 8.8.3.

Theorem 8.8.4. *Let X be a modular operad in $\mathbf{DMV}(\mathbb{C})$. Then $C_*(X; \mathbb{Q})$ is a formal modular operad.*

8.9. Strongly homotopy algebras over a modular operad. Let P be a dg modular operad. Recall ([GeK98]) that a P -algebra is a finite type chain complex V with an inner product B , together with a morphism of modular operads $P \longrightarrow \mathcal{E}[V]$.

We give the following definition. A *strongly homotopy P -algebra*, or *sh P -algebra*, is a finite type chain complex V with an inner product B , together with a morphism $P \longrightarrow \mathcal{E}[V]$ in \mathbf{HoMOp} . By 8.7.2, this is equivalent to giving a homotopy class of morphisms $P_\infty \longrightarrow \mathcal{E}[V]$.

Let (V, B) , and (W, B) be sh P -algebras. A *morphism of sh P -algebras* f is a chain map $f : V \longrightarrow W$ compatible with the inner products and such that the following diagram

$$\begin{array}{ccc}
 & & \mathcal{E}[V] \\
 & \nearrow & \downarrow f_* \\
 P & & \\
 & \searrow & \mathcal{E}[W]
 \end{array}$$

commutes in \mathbf{HoMOp} .

The homotopical invariance is an immediate consequence of the above definitions:

Proposition 8.9.1. *Let (V, B) be a finite type chain complex with an inner product, (W, B) a sh P -algebra, and $f : (V, B) \longrightarrow (W, B)$ a chain map compatible with B and such that $f : V \longrightarrow W$ is a homotopy equivalence. Then (V, B) has a unique structure of sh P -algebra such that f becomes a morphism of sh P -algebras.*

8.10. Application to moduli spaces. Let us apply these results to the modular operad of moduli spaces $\overline{\mathcal{M}}$.

From th. 8.8.4 it follows

Corollary 8.10.1. *$C_*(\overline{\mathcal{M}}; \mathbb{Q})$ is a formal modular operad.*

So, we obtain

Corollary 8.10.2. *Every structure of $H_*(\overline{\mathcal{M}}; \mathbb{Q})$ -algebra lifts to a structure of $sh\ C_*(\overline{\mathcal{M}}; \mathbb{Q})$ -algebra.*

In conclusion, we see that the minimal model $H_*(\overline{\mathcal{M}}; \mathbb{Q})_\infty$ of the modular operad $H_*(\overline{\mathcal{M}}; \mathbb{Q})$ plays an important role in the description of the $sh\ H_*(\overline{\mathcal{M}}; \mathbb{Q})$ -algebras and therefore of the $sh\ C_*(\overline{\mathcal{M}}; \mathbb{Q})$ -algebras. The explicit construction of the modular operad $H_*(\overline{\mathcal{M}}; \mathbb{Q})_\infty$, as in the proof of th. 8.6.3, would require the knowledge of the homology of the moduli spaces and all its relations. We think that a motivic formulation of this minimal modular operad (see [BM]) and the determination of the basic pieces for this building (for instance, a minimal tensor generating family of simple motives for the smaller abelian tensor subcategory where this operad lives) would be a nice variant of Grothendieck’s “Lego-Teichmüller game”.

REFERENCES

- [Ba] W.L. Baily, Jr., *The decomposition theorem for V -manifolds*, Amer. J. Math. **78**(1956), 862–888.
- [Be] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), 601–617.
- [BM] K. Behrend, Y. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), 1–60.
- [Bo] A. Borel, *Linear algebraic groups*, Second enlarged edition, Springer GTM **126** (1991)
- [D] P. Deligne, *Conjecture de Weil. II*, Publ. Math. IHES, **52** (1980), 137–252.
- [DGMS] P. Deligne, P. Griffiths, J.W. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.
- [Ge] E. Getzler, *Operads and moduli spaces of genus 0 Riemann surfaces*, The moduli space of curves (Texel Island, 1994), 199–230, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.
- [GeK94] E. Getzler and M. Kapranov, *Cyclic operads and cyclic homology*, in *Geometry, Topology and Physics for Raoul Bott*, 167–201, Conf. Proc. Lecture Notes Geom. Topology, IV, Internat. Press, Cambridge, MA, 1995.
- [GeK98] E. Getzler and M. Kapranov, *Modular operads*, Compositio Math. **110** (1998), 65–126.
- [GiK] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), 203–272.
- [Gro] A. Grothendieck, *Esquisse d’un programme*, manuscript 1984. Published in *Geometric Galois actions 1*, 5–48. London Math. Soc. Lecture Note Ser., 242. Cambridge Univ. Press, Cambridge, 1997.
- [Hi] V. Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), 3291–3323.
- [HS] S. Halperin and J. Stasheff, *Obstructions to homotopy equivalences*, Adv. Math. **32** (1979), 233–279.
- [KM] I. Kriz and J.P. May *Operads, algebras, modules and motives*, Asterisque **133**, SMF (1995).
- [Ko] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Physics **48** (1999), 35–72.
- [KS] G.M. Kelly and R. Street *Review of the elements of 2-categories*, in *Category Seminar, Sydney 1972/73* Springer LNM **420** (1974), 75–103.
- [KSV] T. Kimura, J. Stasheff and A.A. Voronov, *Homology of moduli of curves and commutative homotopy algebras*, in *The Gelfand Mathematical Seminars, 1993–1995*, Gelfand Math. Sem., Birkhäuser (1996), 151–170.
- [KVZ] T. Kimura, A.A. Voronov and G.J. Zuckerman, *Homotopy Gerstenhaber algebras and topological field theory*, in *Operads: Proceedings of Renaissance Conferences*, Contemp. Math. **202** (1997), 305–333.
- [Man] Y.I. Manin *Frobenius manifolds, quantum cohomology and moduli spaces*, Coll. Publ. AMS **47** (1999).
- [Mas] W.S. Massey *Singular homology theory*, Springer GTM **70** (1980).
- [Mkl] M. Markl, *Models for operads*, Comm. Algebra **24** (1996), 1471–1500.
- [Mor] J.W. Morgan, *The algebraic topology of smooth algebraic varieties*, Publ. Math. IHES **48** (1978), 137–204.
- [MSS] M. Markl, S. Shnider and J. Stasheff, *Operads in Algebra, Topology and Physics*, Surv. and Monog. AMS **96** (2002).
- [R] A. Roig, *Formalizability of DG algebras and morphisms of CDG algebras*, Ill. J. of Math. **38** (1994), 434–451.
- [Sch] L. Schwartz, *Lectures on complex analytic manifolds*, Tata Institute (1955).
- [Su] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. IHES **47** (1977), 269–331.
- [T] D.E. Tamarkin, *Formality of chain operad of small squares*, Preprint math.QA/9809164, September 1998.
- [Wa] W.C. Waterhouse, *Introduction to affine group schemes*, Springer GTM **66** (1979)

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